NEWTON AS AN ORIGINATOR OF POLAR COÖRDINATES

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The name of Newton, indissolubly linked with the calculus, seldom is associated with analytic geometry, a field to which he nevertheless made important contributions. Newton's use of polar coördinates, for example, seems to have been overlooked completely in the historiography of mathematics. The polar coördinate system is ascribed generally [1] to Jacques Bernoulli in 1691 and 1694, although it has been attributed [2] to others as late as Fontana in 1784. It is the purpose here to call attention to an application of polar coördinates made by Newton probably a score of years before the earliest publication of Bernoulli's work.

In the Horsley edition of the *Opera* of Newton there appears a treatise entitled *Artis analyticae specimina vel Geometria analytica* [3] which is essentially the same as the Newtonian *Method of fluxions*, published in 1736 by Colson. The discrepancy in titles—*Geometria analytica* or *Method of fluxions*—conveniently indicates that the work treats of coördinate geometry as well as the calculus. In fact, its analytic form stands in marked contrast to the synthetic style of the *Principia*, which also contained some elements of the calculus. The *Method of fluxions* makes systematic use of coördinates in problems on tangents, curvature, and rectification. Moreover, Newton did not limit himself, as had his predecessors, to a single type of coördinate system. Having shown how to use fluxions in finding tangents to curves given in terms of Cartesian coördinates, oblique as well as rectangular, Newton included some examples illustrating other types. In connection with these he gave, informally, the equivalent of equations of transformation for polar and rectangular coördinates, $xx+yy=tt$ and $tv=y$, where t is the radius vector and v is a line representing the sine of the vectorial angle associated with the point (x, y). Following these exercises, Newton proceeded to give a more definitive account of non-Cartesian systems: "However it may not be foreign from the purpose, if I also shew how the problem may be perform'd, when the curves are refer'd to right lines, after any other manner whatever; so that having the choice of several methods, the easiest and most simple may always be used" [4]. To illustrate his point, Newton suggested eight new types of coördinate system, made up of various combinations of pairs of distances measured radially from given points, or obliquely to given fixed lines, or curvilinearly along arcs of circles. One of the new systems—Newton refers to it as the "Seventh Manner; For Spirals"—is essentially that now known as polar coördinates. Let A be the center and AB a radius of the circle BG (Figure 1), and let D be any point on the curve ADd. Then, designating BG by x and AD by y, the curve ADd is determined by a relationship between x and y. Newton suggested $x^3-ax^2+axy-y^3=0$ as an illustration, and for this curve he determined, from the proportion $\dot{y}:\dot{x}::AD:At$, the polar subtangent AT for a point D. Similarly Newton found the polar subtangents of $y=ax/b$, "which is the equation to the spiral of Archimedes," and the curve $by=xx$; and, he concluded, "thus tangents may be easily drawn to any spirals whatever" [5].

Following the calculation of the radius of curvature for rectangular Cartesian coördinates x and y, $r=\overline{1+zz}\sqrt{1+zz}/\dot{z}$ where $z=\dot{y}$ and fluxions of independent variables are taken as unity, Newton again turned to the corresponding problem in polar coördinates. Using a diagram and a notation similar to those applied in connection with tangent problems—but with the radius AB of the reference circle taken as unity—he derived the result

$$r \sin \psi = \frac{y + yzz}{1 + zz - \dot{z}},$$

where $z=\dot{y}/y$ and ψ is the angle between the tangent and the radius vector

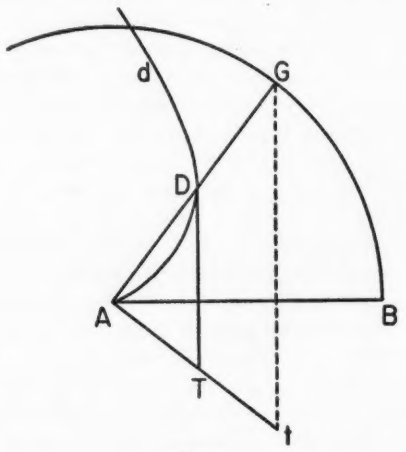

FIG. 1

(fluxions of independent variables again being taken as unity). Newton applied this formula, virtually the same as the modern equivalent, to the spiral of Archimedes and to the curves $ax^2=y^3$ and $ax^2-xy=y^3$. In conclusion he added, "And thus you will easily determine the curvature of any other spirals; or invent rules for any other kinds of curves." That he realized the significance of his use of polar coördinates seems to be implied by his further comment that he here had "made use of a method which is pretty different from the common ways of operation" [6].

The comparison of the parabola with the spiral was a favorite topic of the seventeenth century, and in his treatment of this question, in the *Method of fluxions*, Newton made use of a polar coördinate system yet a third time. Here, however, his scheme differed from that previously presented. The notation, too, was modified, but this may have been done in order to avoid confusion in the simultaneous use of polar and Cartesian coördinates. If D is any point on a curve

ADd, Newton took the coördinates of D as z and v, where z is the radius vector AD and v is the circular arc BD (Figure 2). That is, whereas his earlier coördinates were, in modern notation, $(r, a\theta)$, Newton this time used $(r, r\theta)$. Then if the relation between z and v is given "by means of any equation"; and if a new curve AHh, given in rectangular coördinates $AB=z$ and $BH=y$, is so determined that, for all corresponding positions of D and H, the arc AD is equal to the arc AH; then Newton showed that $\dot{y}=\dot{v}-v\dot{z}/z$, or, if $\dot{z}$ is taken as unity, $\dot{y}=\dot{v}-v/z$. In particular, "if $zz/a=v$ is given as the spiral of Archimedes," then $\dot{v}=2z/a$, and hence $z/a=\dot{y}$ and $zz/(2a)=y$. The lengths of the spirals $z^3=av^2$ and $z\sqrt{a+z}=v\sqrt{c}$ are shown in like manner [7] to correspond respectively to lengths measured along the semi-cubical parabola $z^{3/2}=3a^{1/2}y$ and the curve $(z-2a)\sqrt{ac+cz}=3cy$.

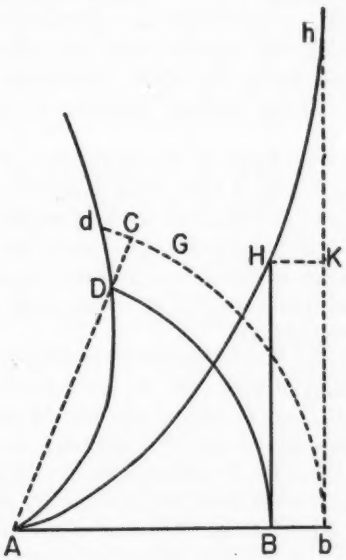

FIG. 2

Evidence indicates [8] that the *Method of fluxions* was composed by 1671, at which time Jacques Bernoulli was in his teens; and there seems to be no reason for suspecting the sections on polar coördinates as later interpolations. The three pertinent passages would appear to be a natural part of the whole; and Horsley, after his editorial examination of three different manuscript copies of the work, apparently saw no reason to question the date or authenticity of this material. It is surprising therefore that this contribution to coördinate geometry should have gone unnoticed so completely that the use of polar coördinates invariably is attributed to others of later periods. Newton is not entitled to priority of publication, for the work appeared posthumously in 1736; but evi-

dence indicates that he was the first one to adopt a system of polar coördinates in strictly analytic form [9]. Moreover, his work in this connection is superior, in flexibility and generality, to any similar proposal to appear during his lifetime.

Priority in the publication of polar coördinates seems to go to Jacques Bernoulli who in the *Acta Eruditorum* of 1691 proposed measuring abscissas along the arc of a fixed circle and ordinates radially along the normals. Three years later, however, he presented in the same journal a system identical, both in conception and notation, with that first proposed by Newton. He used the coördinates y and x, where y is the length of the radius vector of the point and x is the arc cut off by the sides of the vectorial angle on a circle of radius a described about the pole as center. That is, Bernoulli too adopted the coördinates $(r, a\theta)$, whereas in his earlier work he had used a less convenient system equivalent to $(a-r, a\theta)$. Bernoulli, like Newton, was interested primarily in applications of his system to the calculus; and so he also derived a formula for radius of curvature in polar coördinates, [10] and applied it to the spiral of Archimedes, $y=ax:c$.

The polar coördinates of Newton and Bernoulli in 1704 were applied by Varignon [11] to a comparison of the higher parabolas and spirals of Fermat, but no reference was made to Newton's work. Varignon ascribed the idea to Jean Bernoulli and gave to Jacques Bernoulli only the credit for priority of publication. His information in this connection was perhaps not unbiased; and his treatise is tedious and unimaginative in comparison with the work of Newton, at that time still unpublished.

In 1729, two years after Newton's death, Hermann approached polar coordinates from a new point of view. He did not concern himself with spirals, as had Newton, Bernoulli, and Varignon, nor was he chiefly interested in the calculus. He proposed the study of loci "through the relationship which vectorial radii bear to the sine or cosine of the angles of projection, from the consideration of which the properties of curves flow just as elegantly as they are brought out in the usual manner" [12]. That is, Hermann seems first to have thought of polar coördinates as a part of analytic geometry proper. He gave equations for transforming from Cartesian to polar coördinates, and he applied his new system to a number of algebraic curves, including the conics. It should be noted, however, that he did not express his equations specifically in modern form, but wrote them in terms of z, m, and n, where z is the radius vector and m and n are the sine and cosine respectively of the vectorial angle. Moreover, where his predecessors had applied the polar system to spirals alone, Hermann inversely used the scheme exclusively for algebraic curves.

Euler in 1748 seems to have been the first one to combine the points of view of Newton and Hermann. In the influential *Introductio in analysin infinitorum* he devoted a large portion of each of two chapters to polar coördinates, one dealing with algebraic curves and the other with spirals. In the first case [13]

he gave the equations of transformation $x=z\cos\phi$, $y=z\sin\phi$, introducing modern trigonometric symbolism into polar coördinates. He gave general consideration to z as a function of $\sin\phi$ and $\cos\phi$, and he noted in more detail the limaçons $z=b\cos\phi\pm c$ and the conchoids $z=b/\cos\phi\pm c$. In the treatment of transcendental curves Euler adopted a slightly different notion and notation for the independent variable in polar coördinates [14]. Here he studied curves of the form $z=f(s)$, where the argument s is the arc of a unit circle which measures the angle ϕ, feeling, apparently, that coördinates must of necessity denote lengths. In connection with the spiral curves which he drew, Euler made use of the general angle, allowing s to increase indefinitely, both positively and negatively. The spiral of Archimedes therefore appeared, perhaps for the first time, in its dual form [15]. The work of Euler is so thorough and systematic that polar coördinates frequently are attributed to him [16]. Certainly no one after him deserves credit as the inventor of the system. Fontana in 1784 did perhaps supply the name "polar equation" of a curve [17], and he may have been first [18] in studying analytically curves of the form $r=f(\theta, \sin\theta, \cos\theta)$; but one gets the very definite impression that his ideas and manner of treatment were inspired by Euler. It is probably not too much to say that although Newton probably originated polar coördinates, it was the work of Euler which was the decisive factor in making the system a traditional part of elementary analytic geometry. Polar coördinates gradually achieved greater prominence until in 1857 there appeared an entire volume devoted to the analytic geometry of this system in the plane and in space [19]. In 1874 the system was generalized to include elliptic polar coördinates and hyperbolic polar coördinates [20].

It may not be inappropriate to point out here that bipolar coördinates, recently ascribed [21] to Cournot in 1847, also were proposed by Newton. Such a system appeared in the *Method of fluxions* as the "Third Manner" of determining a curve. Here Newton considered [22] the "ellipses of the second order," now known as "ovals of Descartes." In *La géométrie* [23] Descartes had proposed these curves in connection with problems in refraction, but he handled them, as Newton remarked, "in a very prolix manner," without the application of coördinates. Newton therefore seems to have been the originator of bipolar coördinates in the strict sense. Representing by x and y the "subtenses" (or distances) of a variable point from two fixed points (or poles), Newton wrote "their relation" for the ovals as $a+ex/d-y=0$. From this equation he found the ratio of the fluxions, and hence the tangent line. Newton pointed out further that for $a-ex/d-y=0$, a contrary sense is indicated in the construction; and he noted that if $d=e$, the curve becomes a conic section. He closes this topic with the remark that "it would be easy . . . to give more Examples."

Newton's generalizations of the coördinate idea may not be among his greatest contributions to mathematics, but they do entitle him to a larger place in the history of analytic geometry. In this field, as well as in infinitesimal analysis, one may appropriately declare, *Ex ungue leonem.*

References

1. See, *e.g.*, J. L. Coolidge, *A history of geometrical methods* (Oxford, 1940), p. 171. Cf. Florian Cajori, *History of mathematics* 2nd ed., New York, 1931, pp. 221, 224.

2. See, *e.g.*, D. E. Smith, *History of mathematics* (2 vols., New York, c. 1923–1925), II, 324. Cf. Moritz Cantor, *Vorlesungen über Geschichte der Mathematik* (4 vols., Leipzig, 1880–1908), IV, 513.

3. *Opera quae exstant omnia* (5 vols., Londini, 1779–1785), I, 389–518. This appears also (with another title) in Newton's *Opuscula* (3 vols., Lausannae, 1744), I, 29–200.

4. Sir Isaac Newton, *The method of fluxions and infinite series* (transl. by John Colson, London, 1736), p. 51. Cf. *Opera*, I, 435.

5. *Method of fluxions*, p. 56, *Opera*, I, 440.

6. *Method of fluxions*, p. 70, *Opera*, I, 453.

7. *Method of fluxions*, pp. 132–134, *Opera*, I, 511–512.

8. See, *e.g.*, H. G. Zeuthen, *Geschichte der Mathematik im XVI. und XVII. Jahrhundert* (Leipzig, 1903), p. 374; and Coolidge, *op. cit.*, p. 320. Cf. also Newton, *La méthode des fluxions* (Paris, 1740), Buffon's Preface.

9. More or less vague adumbrations of the idea of polar coördinates can, of course, be found in earlier work going back as far as the time of Archimedes' spiral. In early works on perspective the use of concentric circles and radiating lines in problems relating to "deformations" or "anamorphoses" represents a non-analytic application of the polar coordinate idea, much as ideas of latitude and longitude were forerunners of Cartesian coördinates. Some work of James Gregory in 1668 also represents to some extent an anticipation of such a system. See James Gregory, *Tercentenary memorial volume* (ed. by H. W. Turnbull, London, 1939), p. 493.

10. See Jacques Bernoulli, *Opera* (2 vols., Genevae, 1744), I, 432, 578–580. Cf. also a note by Eneström in *Bibliotheca Mathematica* (3), XIII (1912–1913), 76–77.

11. Pierre Varignon, "Nouvelle formation de spirales," *Académie des Sciences, Mémoires*, 1704, pp. 69–131. Cf. *Académie des Sciences, Histoire*, 1704, pp. 47–57. Generalized spirals in polar coordinates appeared also in C. R. Reyneau, *Usage de l'analyse* (vol. II, Paris, 1708), p. 593 and in J. B. Caraccioli, De lineis curvis (Pisis, 1740), p. 161.

12. Jacob Hermann, "Consideratio curvarum in punctum positione datum projectarum, et de affectionibus earum inde pendentibus," *Commentarii Academiae Petropolitanae*, IV (1729), 37–46.

13. Leonhard Euler, *Introductio in analysin infinitorum* (2 vols., Lausannae, 1748), II, 212 ff.

14. *Ibid.*, II, 284 ff.

15. Gino Loria, "Perfectionnements, évolution, metamorphoses du concept de 'coordonnées,'" *Mathematica*, XVIII (1942), 125–145; XX (1944), 1–22, incorrectly ascribes this dual form to Cournot a century later.

16. See, *e.g.*, E. Müller, "Die verschiedenen Koordinatensysteme," *Encyklopädie der mathematischen Wissenschaften*, III (1), 596–770, especially, pp. 656–657. Cf. *Encyclopédie des sciences mathématiques*, III (3), 1, p. 47. See also Loria, *loc. cit.*

17. Gregorio Fontana, "Sopra l'equazione d'una curva," *Memorie di matematica e fisica della società italiana*, II (part 1, 1784), 123–141. See especially p. 128.

18. Gregorio Fontana, *Disquisitiones physico-mathematicae* (Papiae, 1780), pp. 184–185. Fontana used y instead of r, and for θ he took x, where x is the arc of a unit circle measuring the angle θ.

19. J. A. Grunert, *Analytische Geometrie der Ebene und des Raumes für polare Koordinaten* (Greifswald and Leipzig, 1857).

20. C. A. Laisant, "*Essai sur les fonctions hyperboliques*" (Paris, 1874), pp. 71–83.

21. Loria, *op. cit.*, p. 138. Loria here overlooks also the use of bipolar coördinates by L. N. M. Carnot, *Géométrie de position*, (Paris, 1803), p. 469, and J. D. Gergonne, *Annales de mathematiques*, IV, 1813–14, p. 42 f.

22. *Method of fluxions*, p. 54 f, *Opera*, I, 437 f.

23. See *The geometry of Descartes* (translated by D. E. Smith and M. L. Latham, Chicago and London, 1925), p. 114 ff.

THE MOTION OF A SLIDING HORIZONTAL HOOP

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1. Introduction. Two individuals are playing a game with a circular hoop. The rules of the game are as follows:

(a) The circular hoop is to be thrown or launched so that each of its points is always in contact with a horizontal rough floor.*

(b) The hoop must be launched so that its center has a prescribed initial speed $u_0 > 0$.

The objective of the game is to launch the hoop so that it will travel the farthest, distances being measured from the center of the hoop. We shall prove that:

(A) The center of the hoop travels in a straight line, and if the hoop is given a non-zero initial spin, it will stop spinning at the instant the center comes to rest.

(B) The person who throws last can always win, assuming that he has the physical ability to apply the required knowledge.

This mechanical problem owes much of its interest to the fact that although the differential equations governing the motion cannot be solved in terms of elementary functions, interesting qualitative results can be obtained by using methods on a level with a course in Advanced Calculus.

In order to formulate the problem precisely, the following definitions and assumptions are made.

(*i*) Let μ be the coefficient of friction corresponding to the floor and the hoop. The frictional force per unit mass F (a vector) is defined as follows: If a point P of the hoop is moving, then F at P has a direction opposite to that of the velocity of P relative to the floor, and $|F| = \mu g$. If a point P of the hoop is at rest at a time t_1 then the force $F = 0$ at P for the time t_1. The component of the total frictional force on the hoop in a given direction is obtained by the usual process of integration.

2. First results. Let P_0 be the center of the hoop of radius r, and designate the speed of P_0 relative to a fixed coördinate system (ξ, η) by u. Choose an (x, y) coördinate system with center at P_0, so that the positive x-axis has the same direction as that in which P_0 is moving. This is illustrated in the figure, and the (ξ, η) axes are omitted so as not to prejudice the argument. The rotation is assumed to be counterclockwise.

Consider any point P in the hoop with coördinates (x, y), and let θ be the angle between the positive x-axis and the directed line segment P_0P (see figure). The point P relative to P_0 has a velocity V which is perpendicular to P_0P and $|V| = |r\omega|$ where ω is the angular speed of the hoop about P_0. Letting U be the vector velocity of P_0, let ϕ be the angle between U and $U + V$ (see figure), and let $v = r\omega$. Due to assumption (*i*), the force field per unit mass $F(u, v, \theta)$ has the following X and Y components:

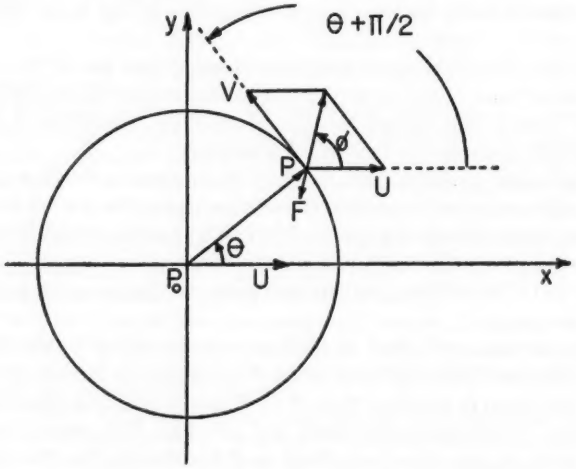
* A circular hoop is a homogeneous one-dimensional circular mass.

$$(1)\qquad \begin{cases} X = Y = 0, & \text{for } (u, v, \theta) = (\delta, \delta, \pi/2) \text{ or for } (u, v, \theta) = (0, 0, \theta), \\ X = -\mu g \cos\phi, \quad Y = -\mu g \sin\phi, & \text{(otherwise),} \end{cases}$$

where δ is an arbitrary real number, where $0 \leqq \theta \leqq 2\pi$, and where $u \geqq 0$, $v \geqq 0$.

Clearly when $(u, v, \theta) \neq (\delta, \delta, \pi/2)$ and when $(u, v) \neq (0, 0)$, by considering the components of U and V, we see that

$$(2)\qquad \cos\phi = \frac{u - v\sin\theta}{\sqrt{u^2 + v^2 - 2uv\sin\theta}}, \qquad \sin\phi = \frac{v\cos\theta}{\sqrt{u^2 + v^2 - 2uv\sin\theta}}.$$



Henceforth we use the notation

$$Q = u^2 + v^2 - 2uv\sin\theta.$$

It is important to note that X and Y in equations (1) are bounded for all (u, v, θ), and that they are continuous in these variables except at $(u, v, \theta) = (\delta, \delta, \pi/2)$ and at $(u, v, \theta) = (0, 0, \theta)$. Since U is tangent to the path of P_0, the normal component a_n of the acceleration of P_0 satisfies the equation

$$2\pi r\sigma a_n = 2\pi r\sigma u^2 K = \int_0^{2\pi} Y\sigma r d\theta, \qquad u \neq 0,$$

where K is the curvature of the path. Since Y is continuous except possibly at $\theta = \pi/2$ when $u \neq 0$, and since $|Y| \leqq \mu g$, equations (1) and (2) imply

$$\int_0^{2\pi} Y d\theta = -\mu g \lim_{\epsilon \to 0} \left[\int_0^{\pi/2-\epsilon} \frac{v\cos\theta}{\sqrt{Q}}\, d\theta + \int_{\pi/2+\epsilon}^{2\pi} \frac{v\cos\theta}{\sqrt{Q}}\, d\theta \right] = 0,$$

as can be verified readily by means of antiderivatives. Thus when $u \neq 0$, the curvature $K=0$. *Hence the center of the hoop moves on a straight line.* This fact can also be seen by considering the forces $F(u, v, \theta)$ and $F(u, v, \pi-\theta)$, $(0 \leqq \theta \leqq 2\pi)$. The Y components of these two forces are equal except for sign, so that the above integral is zero. Hence the x-axis can be chosen parallel to the ξ-axis of the fixed system.

We next derive the differential equations governing the motion of the hoop. By Newton's second law of motion, and by the principle of the moment of momentum, we have

$$
\begin{aligned}
2\pi r\sigma \frac{du}{dt} &= \int_0^{2\pi} X\sigma r\, d\theta, \\
I \frac{d\omega}{dt} &= \int_0^{2\pi} (xY - yX)\sigma r\, d\theta,
\end{aligned} \tag{3}
$$

where $I=2\pi r^3\sigma$, and where $\sigma=$density of the hoop. Since by (1) when $(u, v) \neq (\delta, \delta)$, $\delta \geqq 0$, we have $xY-yX=-\mu g r(\cos\theta \sin\phi - \sin\theta\cos\phi)$, equations (2) imply that

$$
xY - yX = \frac{-\mu g r(v - u\sin\theta)}{\sqrt{Q}}. \tag{4}
$$

Consequently when $(u, v) \neq (\delta, \delta)$, $\delta \geqq 0$, equations (2), (3) and (4) imply that

$$
\begin{aligned}
\frac{du}{dt} &= -a^2 \int_0^{2\pi} \frac{u - v\sin\theta}{\sqrt{Q}}\, d\theta \equiv f(u, v), \\
\frac{dv}{dt} &= -a^2 \int_0^{2\pi} \frac{v - u\sin\theta}{\sqrt{Q}}\, d\theta \equiv g(u, v),
\end{aligned} \tag{5}
$$

where $a^2=\mu g/2\pi$. In the future we shall use (5) to represent the motion also when $(u, v, \theta)=(\delta, \delta, \theta)$, $\delta>0$. However, in this case when $(u, v, \theta)=(\delta, \delta, \pi/2)$ the integrands in (5) are understood to be zero at $\theta=\pi/2$, in accordance with equations (1). In this case we should note that for $(u, v) \neq (0, 0)$

$$
\begin{aligned}
&\int_0^{2\pi} X r\sigma\, d\theta \\
&= -\mu g\sigma r \lim_{\epsilon \to 0} \left[\int_0^{\pi/2-\epsilon} \frac{u - v\sin\theta}{\sqrt{Q}}\, d\theta + \int_{\pi/2+\epsilon}^{2\pi} \frac{u - v\sin\theta}{\sqrt{Q}}\, d\theta \right].
\end{aligned} \tag{6}
$$

A similar statement holds for the second of equations (5).

THEOREM 1. *For any solution* $u(t), v(t)$, *of equations* (5) *for which* $u(0)=u_0>0$, $v(0)=v_0>0$, *there exists a constant* $T(0<T\leqq 2\sqrt{u_0^2+v_0^2}/\mu g)$ *such that*

$$u(t) > 0, \qquad v(t) > 0, \qquad 0 \leqq t < T, \tag{7}$$

$$\lim_{t\to T} u(t) = 0, \qquad \lim_{t\to T} v(t) = 0. \tag{8}$$

Proof. The functions $f(u, v)$, $g(u, v)$ in (5) are continuous functions of u and v since the integrands in these integrals can have at most a single finite discontinuity at $\theta=\pi/2$. Furthermore, since $|f(u, v)| \leqq 2\pi a^2$, $|g(u, v)| \leqq 2\pi a^2$ for all $u>0$, $v>0$, standard existence theorems* imply that through a point (t^*, u^*, v^*), $u^*>0$, $v^*>0$, there passes a solution defined at least on the interval

$$0 \leqq t \leqq \frac{N}{2\pi a^2}, \qquad N = \min\left(\frac{u^*}{2}, \frac{v^*}{2}\right), \tag{9}$$

so that we stay in the first octant of (u, v, t). To prove (7) multiply the first and second of equations (5) by $u(t)$ and $v(t)$, respectively, and add the results so as to obtain

$$\frac{d}{dt}(u^2 + v^2) = -2a^2\int_0^{2\pi} \sqrt{Q}\, d\theta. \tag{10}$$

Since for $u\geqq 0$, $v\geqq 0$,

$$\sqrt{Q} \equiv \sqrt{u^2 + v^2 - 2uv\sin\theta} \geqq \sqrt{u^2 + v^2} \quad \text{for} \quad \pi \leqq \theta \leqq 2\pi$$

we obtain

$$\int_0^{2\pi} \sqrt{Q}\, d\theta \geqq \int_\pi^{2\pi} \sqrt{Q}\, d\theta \geqq \int_\pi^{2\pi} \sqrt{u^2 + v^2}\, d\theta = \pi\sqrt{u^2 + v^2}.$$

Equations (10) then imply that

$$\frac{d}{dt}(u^2 + v^2) \leqq -2a^2\pi\sqrt{u^2 + v^2}, \qquad u \geqq 0, v \geqq 0.$$

Hence on any interval $0\leqq t\leqq t_1 (t_1>0)$ for which $(u^2+v^2)\neq 0$, $u\geqq 0$ $v\geqq 0$, we obtain by integration that

$$0 < \sqrt{u^2 + v^2} \leqq \sqrt{u_0^2 + v_0^2} - a^2\pi t, \qquad 0 \leqq t \leqq t_1. \tag{11}$$

Since $u(0)>0$, $v(0)>0$, (11) implies that $0<t_1\leqq\sqrt{u_0^2+v_0^2}/\pi a^2$.

Equation (11) implies that a *least upper bound* T of the values t for which $u^2(t)+v^2(t)>0$ must exist. Suppose there exists a value t^* with $0<t^*<T$ for which $u(t^*)>0$, $v(t^*)=0$. In this case through the point $[t^*, u(t^*), v(t^*)]$ equa-

* E. Kamke, Differentialgleichungen Reeller Funktionen, Mathematik und ihre Anwendungen (1930), No. 7, pp. 128–130.

tions (5) would have two solutions, the one in question through $[0, u_0, v_0]$ and the solution

$$v_2(t) \equiv 0, \qquad u_2 = u_2(t),$$

where $u_2(t)$ satisfies the equation $du_2/dt = -\mu g$. But when $u \neq v$, $u \neq 0$, $f(u, v)$ and $g(u, v)$ in (5) satisfy the Lipschitz condition* in a neighborhood of (t^*, u^*, v^*), and the solution through this point must be unique. Due to symmetry in (5), we similarly cannot have $u(t^*) = 0$, $v(t^*) > 0$. Hence conditions (7) hold on $0 \leqq t < T$. This same uniqueness theorem implies that we cannot have either

$$\lim_{t \to T} u(t) > 0, \qquad \lim_{t \to T} v(t) = 0, \tag{12}$$

or

$$\lim_{t \to T} u(t) = 0, \qquad \lim_{t \to T} v(t) > 0. \tag{13}$$

To prove (8) in Theorem 1, suppose (8) does not hold. Then the preceding discussion including (12) and (13) together with the fact $|f(u, v)| \leqq 2\pi a^2$, $|g(u, v)| \leqq 2\pi a^2$, $u > 0$, $v > 0$ implies that there exists a constant $M > 0$ such that

$$u(t) > M, \qquad v(t) > M, \qquad 0 \leqq t < T. \tag{14}$$

Choose $t^* = T - M/8\pi a^2$. Then through the point $[t^*, u(t^*), v(t^*)]$ by (9) the solution $u(t)$, $v(t)$ can be extended to be defined on the interval

$$t^* \leqq t \leqq t^* + \frac{N}{2\pi a^2}, \qquad N = \min\left(\frac{u(t^*)}{2}, \frac{v(t^*)}{2}\right).$$

But since by (14), $u(t^*) > M$, $v(t^*) > M$, we have $N \geqq M/2$. Hence $u(t)$, $v(t)$ are defined, and satisfy the relations $u(t) \geqq 0$, $v(t) \geqq 0$, $u^2 + v^2 \neq 0$ on the interval

$$0 \leqq t \leqq t^* + \frac{M}{4\pi a^2} = T - \frac{M}{8\pi a^2} + \frac{M}{4\pi a^2} = T + \frac{M}{8\pi a^2},$$

whence T cannot be the least upper bound of t for which $u^2(t) + v^2(t) > 0$. Consequently the assumption that (8) was false is incorrect, and hence Theorem 1 is proved. Theorem 1 implies the last part of statement (A).

3. A theorem on differential equations. In order to prove statement (B) the following elementary theorem is helpful. Consider the differential equations

$$\frac{du}{dt} = f(u, v, t), \qquad \frac{dv}{dt} = g(u, v, t) \tag{15}$$

where $f(u, v, t)$ and $g(u, v, t)$ are defined in a region $\mathcal{R}(u, v, t)$.

* *Loc. cit.*, p. 141. Since $u(t^*) > 0$, $v(t^*) = 0$ there exists a square (u, v) neighborhood of this point in which $u \neq v$. In this neighborhood the functions $f(u, v)$, $g(u, v)$ have continuous partial derivatives, and hence satisfy the Lipschitz condition.

THEOREM 2. *Suppose the functions* $f(u, v, t)$, $g(u, v, t)$ *are continuous in* $\mathcal{R}(u, v, t)$. *Furthermore suppose*

$$f(u, v_2, t) > f(u, v_1, t) \quad \text{for } v_2 > v_1, \qquad g(u_2, v, t) > g(u_1, v, t) \quad \text{for } u_2 > u_1, \tag{16}$$

for all (u, v, t) *and* (u_1, v_1), (u_2, v_2) *in* $\mathcal{R}$. *Let* $u_1(t)$, $v_1(t)$ *and* $u_2(t)$, $v_2(t)$ *be two solutions of* (15) *which are defined for* $0 \leqq t \leqq T$ *in* $\mathcal{R}$, *and which have no point* (t^*, u^*, v^*) *in common where* $0 \leqq t^* \leqq T$.

Then if

$$u_2(0) = u_1(0), \qquad v_2(0) > v_1(0) \tag{17}$$

holds, it is true that

$$u_2(t) \geqq u_1(t), \qquad v_2(t) \geqq v_1(t), \qquad 0 \leqq t \leqq T. \tag{18}$$

The same conclusion holds if (17) *is replaced by*

$$u_2(0) > u_1(0), \qquad v_2(0) = v_1(0). \tag{19}$$

Proof. Let C be the first connected interval on $0 \leqq t \leqq T$ for which conditions (18) hold. Clearly C is not empty, since $t=0$ is in C. Let t^* be the least upper bound of this bounded connected set. We will prove that $t^* = T$. Suppose $t^* < T$. Since any solution of (15) is continuous, if $u_2(t^*) > u_1(t^*)$, $v_2(t^*) > v_1(t^*)$ then these conditions would still hold on some interval $t^* \leqq t \leqq t^* + \delta$, $\delta > 0$. In this case t^* would not be the least upper bound described. Hence, since, by hypothesis, both equalities in (18) cannot hold on $0 \leqq t < T$ we have either

$$u_2(t^*) = u_1(t^*), \qquad v_2(t^*) > v_1(t^*) \tag{20}$$

or

$$u_2(t^*) > u_1(t^*), \qquad v_2(t^*) = v_1(t^*). \tag{21}$$

Suppose (20) holds. Then since $f(u, v, t)$ is continuous in (u, v, t) and since $u_i(t)$, $v_i(t)$, $(i=1, 2)$ are continuous, equations (16) and (20) imply that there exists a $\delta > 0$ such that

$$f[u_2(t), v_2(t), t] > f[u_1(t), v_1(t), t], \qquad t^* \leqq t \leqq t^* + \delta. \tag{22}$$

Then by equations (15) we have $du_2(t)/dt > du_1(t)/dt$ for $t^* \leqq t \leqq t^* + \delta$. Hence since $u_2(t^*) = u_1(t^*)$, we have

$$u_2(t) \geqq u_1(t), \qquad t^* \leqq t \leqq t^* + \delta.$$

Also since $v_2(t^*) > v_1(t^*)$, from the continuity of these functions there exists a δ_1, $0 < \delta_1 \leqq \delta$ such that $v_2(t) > v_1(t)$ $(t^* \leqq t \leqq t^* + \delta_1)$. Hence

$$u_2(t) \geqq u_1(t), \qquad v_2(t) \geqq v_1(t), \qquad 0 \leqq t \leqq t^* + \delta_1,$$

and both equalities cannot hold for $0 \leqq t < t^* + \delta_1$. Thus t^* is not the least upper

bound as described. A similar result holds if condition (21) holds instead of (20). Consequently we must have $t^* = T$, and Theorem 2 is proved.

4. Proof of statement (B). Statement (B) is restated as follows:

THEOREM 3. *The second man can always win by giving the hoop an initial angular speed absolutely greater than that given by his opponent.*

Proof. Identify $u_1(t)$, $v_1(t)$, $(0 \leqq t \leqq T_1)$, with the first man, and $u_2(t)$, $v_2(t)$, $(0 \leqq t \leqq T_2)$, with the second man. Due to symmetry we can assume without loss of generality that all initial angular speeds are non-negative, so that $v_1(0) \geqq 0$, and so that by hypothesis conditions (17) hold. Note that for the integrands in (5), when

$$(u, v, \theta) \neq (\delta, \delta, \pi/2), \qquad (u, v) \neq (0, 0), \qquad u > 0, v > 0$$

we have

$$(23) \quad \frac{\partial}{\partial v}\left(\frac{u - v \sin\theta}{\sqrt{Q}}\right) = \frac{\partial}{\partial u}\left(\frac{v - u \sin\theta}{\sqrt{Q}}\right) = \frac{-uv\cos^2\theta}{Q^{3/2}} \equiv -h(u, v, \theta) \leqq 0.$$

Since these partial derivatives exist for $0 \leqq \theta < \pi/2$, $\pi/2 < \theta \leqq 2\pi$, applying the theorem of the mean, the first of equations (5) and (23) yield the result,

$$f(u, v_2) - f(u, v_1) = a^2(v_2 - v_1)\left[\int_0^{\pi/2-\epsilon} h[u, \bar{v}(\theta), \theta]d\theta + \int_{\pi/2+\epsilon}^{2\pi} h[u, \bar{v}(\theta), \theta]d\theta\right]$$
$$+ \frac{a^2}{\mu g}\int_{\pi/2-\epsilon}^{\pi/2+\epsilon} [X(u, v_2, \theta) - X(u, v_1, \theta)]d\theta, \qquad v_2 > v_1,$$

where $v_2 > \bar{v}(\theta) > v_1$, and where $h(u, \bar{v}(\theta), \theta)$ is a continuous function of θ for $\theta \neq \pi/2$. Since $h(u, \bar{v}(0), 0) > 0$ $(u > 0)$, and since $|X(u, v, \theta)| \leqq \mu g$ we see that (23) implies that the right side of the above equality is positive when $v_2 > v_1$. Hence the first of conditions (16) is satisfied. Similarly $g(u, v)$ in (5) satisfies the second of conditions (16).

Now let $T = \min [T_1, T_2]$, and let C be the first connected interval of $0 \leqq t \leqq T$ for which $[u_1(t), v_1(t)] \neq [u_2(t), v_2(t)]$. If we denote the least upper bound of C by t^*, condition (17) implies that $t^* > 0$. Since the hypotheses of Theorem 2 are satisfied on $0 \leqq t \leqq t^*$, we must have

$$(24) \qquad u_2(t) \geqq u_1(t), \qquad v_2(t) \geqq v_1(t), \qquad 0 \leqq t \leqq t^*.$$

If $t^* < T$, then we have

$$(25) \qquad u_2(t^*) = u_1(t^*) \neq 0, \qquad v_2(t^*) = v_1(t^*) \neq 0.$$

If the solution of equations (5) through $[u_2(t^*), v_2(t^*), t^*]$ is unique, then (25) is impossible. It is important that the usual Lipschitz condition for equations (5) does not hold, since the partial derivatives in (23) are not bounded for

$0 \leqq \theta < \pi/2$, $\pi/2 < \theta \leqq 2\pi$ when $u = v$, nor do the improper integrals of these functions converge on $0 \leqq \theta \leqq 2\pi$, when $u = v$. We can however prove uniqueness easily as follows.

Equation (10) implies that

$$u^2 + v^2 = u^2(t^*) + v^2(t^*) + 2a^2 \int_t^{t^*} \int_0^{2\pi} \sqrt{Q}\, d\theta dt, \qquad 0 \leqq t \leqq t^*. \tag{26}$$

Hence substituting $u_2(t)$, $v_2(t)$ and $u_1(t)$, $v_1(t)$ in (26), respectively, and subtracting, we obtain

$$(u_2 - u_1)(u_2 + u_1) + (v_2 - v_1)(v_2 + v_1) \leqq \left| 2a^2 \int_t^{t^*} \int_0^{2\pi} |\sqrt{Q_2} - \sqrt{Q_1}|\, d\theta dt \right|$$

where $Q_i = u_i^2 + v_i^2 - 2u_i v_i \sin\theta$, and $u_i = u_i(t)$, $v_i = v_i(t)$, $(i = 1, 2)$, $(0 \leqq t \leqq t^*)$. Observe that when $\theta = \pi/2$, $\sqrt{Q} = |u - v|$. Hence when $\theta = \pi/2$, $|\sqrt{Q_2} - \sqrt{Q_1}| = |\,|u_2 - v_2| - |u_1 - v_1|\,| \leqq |u_2 - v_2 - u_1 + v_1| \leqq |u_2 - u_1| + |v_2 - v_1|$. Furthermore since $\sqrt{Q}$ has bounded derivatives for all $(u, v) \neq (0, 0)$ and for $0 \leqq \theta < \pi/2$, $\pi/2 < \theta \leqq 2\pi$, the theorem of the mean implies that $\sqrt{Q}$ satisfied the Lipschitz condition for all $\theta \neq \pi/2$. Thus $\sqrt{Q}$ satisfies a Lipschitz condition on $0 \leqq \theta \leqq 2\pi$, and hence a constant $K > 0$ exists such that

$$(u_2 - u_1)(u_2 + u_1) + (v_2 - v_1)(v_2 + v_1) \leqq \left| 2a^2 \int_t^{t^*} \int_0^{2\pi} K\{\,|u_2 - u_1| + |v_2 - v_1|\,\} d\theta dt \right| \tag{27}$$

where $0 \leqq t \leqq t^*$. Conditions (7), (24) and (25) imply that

$$u_2(t) \geqq u_1(t) \geqq N > 0, \qquad v_2(t) \geqq v_1(t) \geqq N > 0 \qquad \text{for } 0 \leqq t \leqq t^*.$$

Hence (27) implies that

$$\begin{aligned} 0 \leqq u_2 - u_1 &\leqq \frac{2\pi a^2 K}{N} \left| \int_t^{t^*} \{\,|u_2 - u_1| + |v_2 - v_1|\,\} dt \right|, \\ 0 \leqq v_2 - v_1 &\leqq \frac{2\pi a^2 K}{N} \left| \int_t^{t^*} \{\,|u_2 - u_1| + |v_2 - v_1|\,\} dt \right|, \end{aligned} \tag{28}$$

where $u_i = u_i(t)$, $v_i = v_i(t)$ $(0 \leqq t \leqq t^*, i = 1, 2)$.

By iteration* equations (28) imply that $u_2(t) \equiv u_1(t)$, $v_2(t) \equiv v_1(t)$ $(0 \leqq t \leqq t^*)$. Since this contradicts the hypothesis $v_2(0) > v_1(0) \geqq 0$, the assumption that $t^* < T$ is false, so that (24) holds on $0 \leqq t \leqq T$. If $T_2 < T_1$, then by (24) we would have $u_2(T_2) \geqq u_1(T_2) > 0$, $v_2(T_2) \geqq v_1(T_2) \geqq 0$. Since this contradicts the hypothesis $u_2(T_2) = 0$, $v_2(T_2) = 0$, we have $T_2 \geqq T_1$, so that $T \equiv \min\,[T_1, T_2] = T_1$.

Finally if $ds_2/dt \equiv u_2(t)$ and $ds_1/dt \equiv u_1(t)$, equations (24) imply that

$$\frac{ds_2}{dt} \geqq \frac{ds_1}{dt}, \qquad 0 \leqq t \leqq T_1. \tag{29}$$

* *Loc. cit.*, p. 141.

Since from (22) we see that the equality in (29) cannot hold at all points of the interval, and since $s_2(0)=s_1(0)$, we get $s_2(T_1)>s_1(T_1)$. Since $s_2(T_2)\geqq s_2(T_1)$, we have $s_2(T_2)>s_1(T_1)$, and Theorem 3 is proved.

5. A special case. It should be observed that if $u_2(0)=v_2(0)>0$ then

$$\frac{du_2}{dt}=\frac{dv_2}{dt}=\frac{-\mu g}{2\sqrt{2}\pi}\int_0^{2\pi}\sqrt{1-\sin\theta}\,d\theta=-\frac{2\mu g}{\pi}.$$

It is interesting to compare the distance travelled in this case to the case of pure translation where $u_1(0)=u_2(0)$, $v_1(0)=0$. In the latter case

$$\frac{du_1}{dt}=-\mu g, \qquad v_1(t)\equiv 0.$$

Hence since $u_2(0)=u_1(0)$, we get

$$s_2(T_2)=\frac{u_2^2(0)\pi}{4\mu g}, \qquad s_1(T_1)=\frac{u_2^2(0)}{2\mu g}.$$

Thus $s_2(T_2)=(\pi/2)s_1(T_1)=(1.57+)s_1(T_1)$, which illustrates the diminishing effect of friction due to rotation of the hoop.

ON BOSE NUMBERS

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N. C. Bose Majumdar has given a method* of writing the repetend of the recurring decimal equivalent to the fraction $1/n$, n being a positive integer prime to 10. A brief description of this method is presented before an explanation of it is given.

Taking those fractions whose numerator is 1 and whose denominator does not have a factor 2 or 5, he divided them into four groups according as they have for the last digit in the denominator 1, 3, 7, or 9. He defined the End Number e as the smallest positive integer which when multiplied by the denominator gives a number ending in 9. For each group of such fractions he gave a rule for writing down the "Bose Number b." In general, however, his Bose Number is given by the equation $n\cdot e=10b-1$.

Bose's method consisted in writing the End Number e, multiplying it by the Bose Number b, and placing the last digit of the product before e, calling it e_2 and carrying over the remaining digits, multiplying e_2 by b and adding the

* Ravenshavian, A magazine of Ravenshaw College, Cuttack, Orissa, India; vol. XXVI, no. 2, April, 1942.

number carried over and writing the last digit of this sum as e_3, and so on, until the digits recur.

EXAMPLE. For 1/21, n is 21; $21 \cdot 9$ is $189 = 19 \cdot 10 - 1$, so that e is 9 and b is 19.

Write down the End Number 9 first, multiply this 9 by the Bose Number 19 and obtain 171. Place 1 before 9 and carry 17; multiply this 1 by the Bose Number 19 and add 17, obtaining 36. Place 6 before 1 and carry 3; multiply this 6 by 19 and adding 3, we get 117. Place 7 before 6 and carry 11, and so on. Continue the process until the numbers recur. For 1/21 we arrive at $.\dot{0}47619\dot{0}$.

Bose Majumdar did not give any mathematical explanation for his method. It is the purpose of this paper to give a proof for the soundness of this method of writing down the repetend of the recurring decimal.

The relation between n, e, and b is given by the equation

$$n \cdot e = 10b - 1. \tag{1}$$

Suppose r is the repetend of k digits of the recurring decimal for $1/n$, so that,

$$r = e_1 + 10e_2 + 10^2e_3 + \cdots + 10^{k-1}e_k. \tag{2}$$

$$1/n = r/(10^k - 1). \tag{3}$$

Then Bose's method indicates that the successive digits of r (counting from the right toward the left) may be obtained as follows;

The first digit e_1 is e.

The first two digits of $e_1 + 10be_1$ give the first two digits of r.

The first three digits of $e_1 + 10be_1 + 10^2be_2$ give the first three digits of r; and, in general, the first i digits of $e_1 + 10be_1 + 10^2be_2 + 10^3be_3 + \cdots + 10^{i-1}be_{i-1}$ are the first i digits of r, $i = 2, 3, \cdots, k$.

EXAMPLE. $n = 21$, $21 \cdot 9 = 189 = 10 \cdot 19 - 1$; $e = e_1 = 9$, $b = 19$. Then

$10 \cdot 19 \cdot 9$ is	1710	
	9	
	1719	showing e_2 is 1.
$10^2 \cdot 19 \cdot 1$ is	1900	
	1710	
	9	
	3619	showing e_3 is 6.
$10^3 \cdot 19 \cdot 6$ is	114000	
	1900	
	1710	
	9	
	117619	showing e_4 is 7.

This can be condensed into the form

$$\begin{aligned} 7600000 &= 10^5 \cdot 19 \cdot 4 \\ 1330000 &= 10^4 \cdot 19 \cdot 7 \\ 114000 &= 10^3 \cdot 19 \cdot 6 \\ 1900 &= 10^2 \cdot 19 \cdot 1 \\ 1710 &= 10 \cdot 19 \cdot 9 \\ 9 & \\ \hline 047619 &= r, \text{ so } 1/21 \text{ is } .\dot{0}47619\dot{0}. \end{aligned}$$

To prove the soundness of the procedure, we must prove* that

(4) $e_1 + 10be_1 + 10^2be_2 + \cdots + 10^{i-1}be_{i-1} \equiv r, \bmod 10^i \quad \text{for } i = 2, 3, \cdots, k.$

Proof. Eliminating n from (1) and (3) with $e_1 = e$, gives

(5) $$(10b - 1)r = (10^k - 1)e_1.$$

Then, using (2), we have

$$10be_1 + 10^2be_2 + \cdots + 10^kbe_k - r = 10^ke_1 - e_1.$$

Thus,

$$e_1 + 10be_1 + 10^2be_2 + \cdots + 10^{i-1}be_{i-1} - r = 10^ke_1 - 10^ibe_i - \cdots - 10^kbe_k;$$

or

$$e_1 + 10be_1 + 10^2be_2 + \cdots + 10^{i-1}be_{i-1} \equiv r, \bmod 10^i; \qquad \text{for } i = 2, 3, \cdots, k.$$

In cases where the numerator is not unity, but is less than the denominator, the End Number is first multiplied by the numerator to get e_1 and then the procedure just outlined is followed.

EXAMPLE. For $3/7$, $7 \cdot 7 = 49 = 10 \cdot 5 - 1$ so that e is 7 and b is 5. The product of 7 and 3 is 21, so that the End Number for $3/7$ is written as 1, carrying 2; this 1 is multiplied by the Bose Number 5, and 2 is added, thereby obtaining 7 which is e_2. The method that is used in the case where the numerator is 1 then gives $.\dot{4}28571\dot{4}$ for $3/7$.

If the numerator is greater than the denominator, it is first broken down into the whole number plus a fraction (less than 1) which is then used in writing the recurring decimal. In cases where the denominator has a factor 2 or 5, it is written as $1/10^m$ times a fraction whose denominator does not have a factor 2 or 5, for which the equivalent decimal is written in the same manner as above.

EXAMPLE $131/52 = 131/4 \cdot 13 = (25 \cdot 131)/(10^2 \cdot 13)$ or $(1/10^2) \cdot (3275/13) = (1/10)^2 \cdot (251 + 12/13)$. Now, $13 \cdot 3$ is 39 or $10 \cdot 4 - 1$. Thus e is 3, and b is 4. For $12/13$ we get $.\dot{9}2307\dot{6}$. So $131/52$ is $(1/10)^2 \cdot (251 \cdot \dot{9}2307\dot{6})$ or $2.51\dot{9}2307\dot{6}$.

* The author acknowledges some suggestions that he received from Professor W. B. Carver in connection with this proof.

A PROGRAM OF INFORMATION FOR PROSPECTIVE COLLEGE STUDENTS

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Members of the Michigan Section of the Association have been concerned for some years over the increasing number of students coming to college and wishing to enroll in courses and curricula for which they did not have the necessary prerequisites in high school mathematics. This concern was intensified by the adoption of the Michigan College agreement, which under certain conditions allows high school graduates to be admitted to college without regard to the pattern of courses taken in the high school.

Early in 1947, a committee consisting of Professor H. W. Alexander (Adrian College), Professor P. S. Jones (University of Michigan), and Professor C. C. Richtmeyer (Central Michigan College), was appointed to consider the problem and to make specific recommendations to the Association at the Spring meeting. At this meeting, the committee proposed the following recommendations which were approved by the Association:

1. That the Association authorize the preparation of a pamphlet pointing out to high school counselors and students the mathematical prerequisites necessary for admission to various college courses and curricula, and the difficulties resulting from the deferment of such mathematical preparation until the time for college entrance. In compiling the information for this pamphlet, each college or school of college grade in Michigan should be asked to cooperate by furnishing a list of curricula and courses together with the mathematical requirements for each.

   Another possible means of disseminating information would be the preparation of an attractive poster depicting the areas of collegiate study requiring previous mathematical study.

2. That the officers of this Association or someone designated by them recommend to all college and university registrars, or officers in charge of preparation of college catalogues, that each curriculum specifically state the mathematical prerequisites necessary for entrance upon that curriculum, and that the general statements of entrance requirements carry a reference to these prerequisites. It is further recommended that all collegiate heads of departments of mathematics be urged to work for the insertion of such a statement in their respective college catalogues.

3. That we seek the cooperation of other mathematical organizations in carrying out this project and that such publications as the *Michigan Education Journal* be asked to cooperate in the dissemination of the information.

4. That the Association express its willingness to appropriate sufficient funds to carry out such of the foregoing recommendations as it approves.

Under the direction of Professor Jones, a survey was made of all Michigan Colleges to find out what courses and curricula require high school mathematics

as prerequisite. Professor Alexander undertook the preparation of a chart or poster which would depict the areas of collegiate study requiring high school mathematics. The writer undertook the project of urging the colleges to include in their respective catalogues specific statements regarding mathematical prerequisites for the various courses and curricula. In connection with the latter project, letters were sent to all persons in charge of editing college catalogues urging them to include specific references to mathematical prerequisites in the next edition of their catalogue. Letters were also sent to each college mathematics department head, asking him to cooperate in seeing that this was done. Resolutions regarding this item were presented to the Michigan College Association and adopted by that body.

The results of Professor Jones' survey were incorporated into a pamphlet entitled, *A Mathematics Student—To Be or Not To Be*, which has been mailed recently to all high school principals and all college mathematics department heads in the state. The chart prepared under the direction of Professor Alexander was incorporated as the frontispiece of the pamphlet, and was also printed separately to be used as a bulletin board poster. Copies of the poster were also sent to all high schools.

The pamphlet and chart have created considerable interest both in Michigan and outside the state. It is hoped that they may serve a real guidance function for prospective college entrants.

MATHEMATICAL NOTES

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Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

NOTES ON THE GRAEFFE METHOD OF ROOT SQUARING

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1. Introduction. In this paper a method is explained for removing some of the defects of the Graeffe method [1, 2, 3, 4] of root squaring for the determination of the zeros of a polynomial. The Graeffe method as outlined in [1], for example, becomes awkward [4] when two or more zeros have identical or nearly identical moduli. By the procedure of [1] the process of root squaring is first carried out m times, say, yielding the polynomials

$$\sum_{i=0}^{n} a_{im} x^{n-i} = 0,$$

the zeros of which are the (2^m)th powers of those of the original poly-

nomial. Then, if all zeros are real and single, the values $A_{im}=a_{im}/a_{i-1,m}$, $(i=1, 2, 3 \cdots, n)$, are computed, that is, each coefficient is divided by that preceding it, and the (2^m)th roots of these ratios taken, yielding the roots of the original equation.

If double zeros occur, however, it is necessary to skip a coefficient in this dividing process; that is, one should divide by the coefficient before the one immediately preceeding [4]. The (2×2^m)th root is then taken of the resulting A. In general for an s-fold root, $s-1$ coefficients should be skipped and the $(s\times 2^m)$th root taken. Hence it is imperative to know the degree of multiplicity of each root before computing the A's.

2. Auxiliary Table. A simple way of determining the multiplicity of the roots and also determining whether they are real or complex is to construct an auxiliary table composed of the elements

$$\rho_{im} = \frac{a_{i,m-1}}{a_{im}} \Big| a_{i,m-1} \Big|.$$

The A's can then be determined by considering only those coefficients for which the corresponding ρ's are unity. Dividing each such coefficient by the preceding such coefficient yields the A's from which the zeros may then be obtained, remembering that if $s-1$ coefficients are skipped when computing A_{im}, so that $A_{im}=a_{im}/a_{i-s,m}$, then the $(s\times 2^m)$th root of A_{im} should be taken.

If a zero is real and s-fold the ρ's in the auxiliary table will converge to the binomial coefficients of $(1+1)^s$, it being easy thereby to recognize the presence of real roots. Also if s real roots are very nearly identical, then the corresponding ρ's will differ but slightly from the above binomial coefficients. If s complex roots (i.e., $s/2$ pairs) have identical moduli then the $s-1$ corresponding ρ's between those converging to unity will not converge to any value—it being possible by this means to detect complex roots.

The above can be easily proved by methods similar to those used in [1]. Briefly, letting the negatives of the roots (or Encke roots) of a polynomial be a, b, c, and so on, and assuming that $a \gg b \gg c$, and so on, then it is possible to write

$$x^n + [a]x^{n-1} + [ab]x^{n-2} + [abc]x^{n-3} + \cdots = 0, \tag{1}$$

where $[a]$ denotes $a+b+c+\cdots$, and $[ab]$ denotes $ab+bc+ac+\cdots$, and so on.

Because of the large differences in size, (1) can be written

$$x^n + ax^{n-1} + abx^{n-2} + abcx^{n-3} + \cdots = 0. \tag{2}$$

Suppose now that a is s-fold. Equation (2) becomes

$$x^n + C_1^s a x^{n-1} + C_2^s a^2 x^{n-2} + C_3^s a^3 x^{n-3} + \cdots = 0. \tag{3}$$

The squaring process then gives:

$$x^n + C_1^s a^2 x^{n-1} + C_2^s a^4 x^{n-2} + \cdots = 0, \tag{4}$$

the corresponding ρ's becoming $\rho_i = C_i^s a^i / C_{i-1}^s a^{2i} \left| C_i^s a^i \right| = C_i^s$. We would proceed similarly if b were s-fold, and so on.

3. Complex Roots. A useful application of Descartes rule of signs occurs with the Graeffe method. With the first squaring yielding the polynomial p_2, say, all real roots of the original polynomial p_1 go over into real positive roots in in p_2. Hence if k is equal to the number of changes in sign in p_2, then $n-k$ sets a lower limit to the number of complex roots present in p_1. Once the moduli of these roots are determined the corresponding angles can be obtained by the method of [5].

4. Illustrative Example. Given the polynomial

$$x^5 - 3x^4 + 10x^3 - 68x^2 + 168x - 128 = 0,$$

determine the moduli r and the multiplicity s of all roots. Computations are shown in Table I. In the notation used, 123^7 is written for 1.23×10^7. The auxiliary table is computed only for the last three squarings since early convergence of the ρ's is unlikely. After the second squaring, with $m=1$, it can be observed that since p_2 has 3 changes in sign there cannot be fewer than $5-3=2$ complex zeros. With regard to the method, an easy way to compute the successive sets of coefficients is to write those last obtained backward on a card spacing them as in the forward arrangement but with every other sign after the first altered. By matching coefficients in the forward and backward arrangements, we compute the new coefficient in any line by accumulating the products of figures which are then opposite, noting that such products are symmetrical about the middle squared term. By this procedure the signs of alternate sets of coefficients will differ, if n is odd, from these obtained from the formulas of [1]. This does not affect the value of the moduli, hence is not significant. This method is used in the following table.

TABLE I

m	a_{0m}	a_{1m}	a_{2m}	a_{3m}	a_{4m}	a_{5m}
0	1	-3^0	$+10^1$	-68^1	$+168^2$	-128^2
1	1	$+11^1$	$+28^1$	-2032^3	$+10816^4$	-16384^4
2	1	-65^1	$+6712^4$	-316288^6	$+5040128^7$	-26843546^8
3	1	$+130015^8$	$+41947226^9$	-32728386^{12}	$+84223073^{14}$	-72057596^{16}
4	1	-85144550^9	$+18448418^{19}$	-36268868^{24}	$+23768684^{29}$	-51922971^{33}
5	1	-35599108^{19}	$+34028236^{38}$	-43845039^{48}	$+18831286^{58}$	-26959949^{67}
6	1	-58673177^{38}	$+11579208^{77}$	-64079656^{96}	$+11820533^{116}$	-72683885^{134}
s			2 (complex)			3
A			11579208^{77}			62771033^{67}
$\log A$			77.0636786			57.7977592
$s2^m$			128			192
$\log r$			.60205999			.301029996
r			4.0			2.0

AUXILIARY TABLE

3:4	1	−1.985318	0.953778	2.953352	2.984400	1.0
4:5	1	+2.036454	1.000182	3.000182	3.000062	1.0
5:6	1	+2.159925	1.0	2.999996	3.000011	1.0

References

1. E. T. Whittaker and G. Robinson, The Calculus of Observations, Blackie & Son, 2nd edition, p. 106.

2. C. Runge and H. Konig, Numerisches Rechnen, Julius Springer, p. 164.

3. Th. v. Kármán and T. A. Biot, Mathematical Methods in Engineering, McGraw-Hill, 1st edition, p. 194.

4. E. T. Whittaker and G. Robinson, The Calculus of Observations, Blackie & Son, 2nd edition, p. 117.

5. G. C. Best, The Determination of the Complex Zeros of a Polynomial, this MONTHLY, vol. 54, 1947, p. 269.

NOTE ON THE APPROXIMATE SOLUTION OF AN OBLIQUE TRIANGLE WITHOUT TABLES

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If the three sides a, b, c of an oblique triangle ABC are given, the smallest angle A may be obtained to a very good approximation by the formula

$$A = \frac{6\sqrt{(s-b)(s-c)}}{2\sqrt{bc} + \sqrt{s(s-a)}}, \quad \text{where} \quad s = \frac{a+b+c}{2}. \tag{1}$$

Formula (1) may be derived by proceeding in a manner similar to that used for obtaining an analogous formula for the right triangle.* It may be noted that

$$2\csc\frac{A}{2} + \cot\frac{A}{2} = \frac{6}{A} + \frac{A^3}{480} + \frac{A^7}{16,128} + \cdots. \tag{2}$$

As an approximation, for small values of A, we may neglect all but the first term of the right hand member of (2). We have then

$$\frac{A}{2} \cong \frac{3}{\dfrac{2}{\sin\dfrac{A}{2}} + \dfrac{1}{\tan\dfrac{A}{2}}}. \tag{3}$$

The right hand member of (3) is the harmonic mean of $\sin A/2$ and $\tan A/2$, weighted two to one.

Using the functions of the half angles for oblique triangles, we obtain

* Solving a Right Triangle Without Tables. J. S. Frame, this MONTHLY, Vol. 50, 1943, pp. 622. See also article by R. A. Johnson, this MONTHLY, Vol. 27, 1920, p. 365, Bibliography, p. 366.

$$\frac{6}{A} \cong 2\sqrt{\frac{bc}{(s-b)(s-c)}} + \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

or

$$A \cong \frac{6\sqrt{(s-b)(s-c)}}{2\sqrt{bc} + \sqrt{s(s-a)}}.$$

The value of an angle obtained by formula (1) is correct to 5 decimal places for angles less than 30° The following table shows the error, E, for given angles.

A (degrees)	0°	10°	20°	30°	40°	50°	60°
E (minutes)	0′	.0′	.0′	.0′	.2′	1.0′	1.5′

where the error is given by the formula

$$(5) \quad \text{Error (radians)} = A - \frac{6\sin\frac{A}{2}}{2+\cos\frac{A}{2}} = \frac{A^5}{2880} + \frac{A^7}{96{,}768} + \cdots \cong .00035A^5.$$

This error is about 1/9 the corresponding error if the arithmetic mean had been used instead of the harmonic mean.

As an example, we may solve the triangle with sides 7, 9 and 14. By use of formula (1)

$$A = \frac{6\sqrt{6}}{6\sqrt{14} + 2\sqrt{30}} = .43997 \text{ radians or } 25°12.5'.$$

Likewise, $B = .57948$ radians or 33°12.1′.

For a better approximation for C, replace C (obtuse) by $\pi - C$ in formula (1). Hence

$$(4) \quad \pi - C = \frac{6\sqrt{s(s-c)}}{2\sqrt{ab} + \sqrt{(s-a)(s-b)}}.$$

Substituting in formula (4) we obtain

$$\pi - C = 1.01908 \text{ radians or } 58°23.3'$$

so that

$$C = 121°36.7'.$$

Formula (1) gives answers for A and B which are as accurate as those obtained by half-angle formulas and a five place logarithm table. Formula (4) gives an answer for C which is as accurate as if it were obtained by a four place logarithm table. Formulas (1) or (4) should be valuable for long hand computation without tables.

NOTE ON A PERIODIC CONTINUED FRACTION

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If the complex number a is not a real number less than $-\frac{1}{4}$, then the continued fraction

$$1+\cfrac{a}{1+\cfrac{a}{1+\cdot}}\cdot \tag{1}$$

converges to the numerically larger root of the equation $x^2-x-a=0$.* We observe that if Newton's formula

$$x_{p+1}=x_p-f(x_p)/f'(x_p), \qquad p=0,1,2,\cdots, \tag{2}$$

is used to compute a root of $f(x)=x^2-x-a$, starting with x_0 equal to one of the approximants $f_1=1$, $f_2=1+a$, $f_3=(1+2a)/(1+a)$, $\cdots$ of (1), say with f_n, then $x_1=f_{2n}$, $x_2=f_{4n}$, $x_3=f_{8n}$, $\cdots$. The same phenomenon occurs if, instead of (2), we use Frame's† modification of Newton's formula, namely,

$$x_{p+1}=x_p-\frac{f(x_p)}{f'(x_p)-\dfrac{f(x_p)f''(x_p)}{2f'(x_p)}}, \qquad p=0,1,2,\cdots, \tag{3}$$

except that here $x_1=f_{3n}$, $x_2=f_{9n}$, $x_3=f_{27n}$, $\cdots$. The following theorem includes both these facts.

THEOREM. *Let $U_1(x)$, $U_2(x)$, $U_3(x)$, $\cdots$ denote the sequence of approximants of the periodic continued fraction*

$$x-\cfrac{x^2-x-a}{2x-1-\cfrac{x^2-x-a}{2x-1-\cdot}}\cdot\cdot, \tag{4}$$

and let $f_1, f_2, f_3, \cdots$ be the sequence of approximants of (1). *Then*

$$U_m(f_n)=f_{mn}, \qquad m,n=1,2,3,\cdots. \tag{5}$$

Proof. Inasmuch as $f_n=U_n(1)$, (5) is a consequence of the following more general formula:

$$U_m[U_n(x)]=U_{mn}(x), \qquad m,n=1,2,3,\cdots. \tag{6}$$

In the special case $a=-1/4$, we readily find that

* See, for instance, H. S. Wall, *Analytic Theory of Continued Fractions*, D. Van Nostrand Co., 1948, p. 39.

† This MONTHLY, vol. 51, 1944, pp. 36–38.

$$U_n(x) = \frac{x}{n} + \frac{n-1}{2n},$$

and therefore (6) holds.

If $4a+1 \neq 0$, then the roots r and s of the equation $x^2-x-a=0$ are distinct. We make the transformation

$$\zeta = \frac{x-r}{x-s}, \qquad x = \frac{s\zeta - r}{\zeta - 1}, \tag{7}$$

throwing r to the origin and s to ∞. Under this substitution, the continued fraction (4) becomes

$$\frac{s\zeta - r}{\zeta - 1} - \frac{s-r}{\zeta-1} \cdot \cfrac{\zeta}{\zeta + 1 - \cfrac{\zeta}{\zeta + 1 - \cfrac{\zeta}{\zeta + 1 - \cdot}}},$$

from which we find that

$$U_n(x) = \frac{s\zeta - r}{\zeta - 1} - \frac{s-r}{\zeta-1} \cdot \frac{\zeta(\zeta^{n-1}-1)}{\zeta^n - 1},$$

or

$$U_n(x) = \frac{s\zeta^n - r}{\zeta^n - 1}, \qquad \zeta^n = \frac{U_n(x) - r}{U_n(x) - s}. \tag{8}$$

Consequently, by (7),

$$U_m[U_n(x)] = U_m\left[U_n\left(\frac{s\zeta - r}{\zeta - 1}\right)\right] = U_m\left(\frac{s\zeta^n - r}{\zeta^n - 1}\right) = \frac{s\zeta^{mn} - r}{\zeta^{mn} - 1} = U_{mn}(x),$$

and the theorem is established.

Let the notation be chosen so that $|r| \geqq |s|$, and suppose that $|r| > |s|$. Then

$$\left|\frac{s}{r}\right| = \left|\frac{1-r}{r}\right| < 1, \qquad R(r) > \frac{1}{2}.$$

By (8), $U_n(x) \to r$ or $U_n(x) \to s$, as $n \to \infty$, according as $|\zeta| < 1$, or $|\zeta| > 1$, respectively, i.e., according as $|x-r| < |x-s|$ or $|x-r| > |x-s|$. Thus, if L is the perpendicular bisector of the line-segment from r to s, and H_r and H_s are the open half-planes bounded by L, containing r and s, respectively, then $U_n(x)$ converges, for $n \to \infty$, to r or to s, according as x is in H_r or in H_s, respectively. We note that L passes through the point $1/2$. In particular, $U_n(1) \to r$, $U_n(0) \to s$, i.e., $f_n \to r$, $-a/f_n \to s$.

CLASSROOM NOTES

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CONTINUED FRACTIONS AND MATRICES

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1. Introduction. The closest rational approximation P/Q (P, Q integers) to a given positive real number x, subject to the condition that the denominator Q shall not exceed a given positive integer N, is most easily found by means of continued fractions. For example, successive approximations to π obtained by this method are 3, 22/7, 333/106, 355/113, $\cdots$. Continued fractions may also be used to facilitate the solution of diophantine equations.

It is our purpose to show how the principal theorems in the theory of continued fractions, and the actual computation of successive convergents to a continued fraction can be presented quite simply by the use of two-rowed matrices and their determinants. To a student familiar with the multiplication of two-rowed matrices a good introduction to continued fractions can be presented in an hour's lecture. For a student who has just been introduced to matrices, their use in connection with continued fractions provides an easy application at the elementary level.

2. Simple continued fractions. For a given positive real x let the integral part be a_1 and the remainder r_1, with $0 \leqq r_1 < 1$. We define successively the positive integers $a_2, a_3, \cdots, a_n$, (called partial denominators) and the remainders $r_2, r_3, \cdots r_n$ so that

$$(1) \qquad x = a_1 + r_1, \qquad \frac{1}{r_1} = a_2 + r_2, \cdots, \frac{1}{r_{n-1}} = a_n + r_n, \qquad 0 \leqq r_k < 1.$$

and we write

$$(2) \qquad x = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n + r_n}}}}}$$

or

$$(2') \qquad x = a_1 + \frac{1}{a_2 +} \quad \frac{1}{a_3 +} \quad \cdots \frac{1}{a_{n-1} +} \quad \frac{1}{a_n + r_n}.$$

After simplifying the complex fraction (2) and collecting coefficients of r_n, we have

$$x = \frac{P_n + r_n S_n}{Q_n + r_n T_n}, \qquad \text{where} \quad M_n = \begin{pmatrix} P_n & S_n \\ Q_n & T_n \end{pmatrix} \tag{3}$$

is an integral matrix, and where, for the case $n=1$, we have from (1)

$$M_1 = \begin{pmatrix} P_1 & S_1 \\ Q_1 & T_1 \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}.$$

It will be convenient to denote the nth denominator in (3) by D_n;

$$D_n = Q_n + r_n T_n \tag{4}$$

To obtain a recursion formula for the P's and Q's, we replace n by $n-1$ in equation (3), divide numerator and denominator by r_{n-1}, and apply (1).

$$x = \frac{P_{n-1} + r_{n-1}S_{n-1}}{Q_{n-1} + r_{n-1}T_{n-1}} = \frac{P_{n-1}(a_n + r_n) + S_{n-1}}{Q_{n-1}(a_n + r_n) + T_{n-1}} = \frac{(a_n P_{n-1} + S_{n-1}) + r_n P_{n-1}}{(a_n Q_{n-1} + T_{n-1}) + r_n Q_{n-1}}. \tag{5}$$

On comparing coefficients of r_n in (3) and (5) we have

$$\begin{aligned} S_n &= P_{n-1}, & P_n &= a_n P_{n-1} + S_{n-1} \\ T_n &= Q_{n-1}, & Q_n &= a_n Q_{n-1} + T_{n-1}. \end{aligned} \tag{6}$$

Hence,

$$\begin{aligned} P_n &= a_n P_{n-1} + P_{n-2} \\ Q_n &= a_n Q_{n-1} + Q_{n-2}. \end{aligned} \tag{7}$$

Also from (4) and (5) the ratio D_{n-1}/D_n is seen to be r_{n-1}, and $D_1 = 1$, so

$$1/D_n = r_1 r_2 \cdots r_{n-1}, \tag{8}$$

provided that none of the r's are 0. The set of equations (7) can most easily be written in matrix form

$$M_n = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = M_{n-1} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{9}$$

By induction we then obtain the fundamental relation

$$M_n = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}. \tag{10}$$

3. Theorems on continued fractions. Since the determinant of the matrix M_n is the product of the determinants of its factors, we obtain from (10) the important relations

$$\begin{vmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{vmatrix} = (-1)^n \tag{11}$$

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^n}{Q_nQ_{n-1}} \tag{12}$$

$$x - \frac{P_n}{Q_n} = \frac{P_n + r_nP_{n-1}}{Q_n + r_nQ_{n-1}} - \frac{P_n}{Q_n} = \frac{r_n(-1)^{n-1}}{Q_nD_n} = \frac{(-1)^{n-1}}{Q_nD_{n+1}}. \tag{13}$$

THEOREM I. *The rational fractions P_n/Q_n, called the "convergents" of x, are alternately less and greater than x, and the difference of successive convergents is the reciprocal of the product of their denominators. Since this product becomes infinite with n, the differences defined by* (12) *approach zero and alternate in sign as n increases, so the sequence P_n/Q_n converges. Its limit is x.*

The proof of these statements follows directly from (13) and (12).

Equation (3) may be replaced by the matrix equation

$$\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} \begin{pmatrix} 1/D_n \\ 1/D_{n+1} \end{pmatrix}. \tag{14}$$

Solving for r_n, which is the ratio D_n/D_{n+1}, we have

$$r_n = -\frac{P_n - Q_nx}{P_{n-1} - Q_{n-1}x} \tag{15}$$

$$a_{n+1} + r_{n+1} = \frac{Q_{n-1}x - P_{n-1}}{-Q_nx + P_n}. \tag{16}$$

Equation (16) makes it possible to compute a_{n+1} directly if the matrix M_n is known. The next matrix M_{n+1} is then obtained from (10). For example, in the continued fraction expansion for π the partial denominator a_5 is the largest integer in $(106\pi - 333)/(-113\pi + 355) = 292.+$

THEOREM II. *A periodic continued fraction represents a root of a quadratic equation.*

Proof. If two different remainders are equal, equate them, using (15), and solve for x.

THEOREM III. *The rational fraction P_n/Q_n defined by* (1) *and* (10) *differs from x by not more than $1/Q_nQ_{n+1}$, and it approximates the real number x more closely than does any other rational fraction with a denominator not exceeding Q_n.*

Proof. The difference (13) is equal in absolute value to $1/Q_nD_{n+1}$, which by (4) is certainly less than $1/Q_nQ_{n+1}$. Suppose there were two integers A and B, with $0 < B < Q_n$, such that the fraction A/B were closer to x than P_n/Q_n. Then we should have

$$-\frac{1}{Q_nD_{n+1}} < (-1)^n\left(x-\frac{A}{B}\right) < \frac{1}{Q_nD_{n+1}}. \tag{17}$$

Adding $1/Q_nD_{n+1}$ to each term and replacing x by its value in (13) gives

$$0 < (-1)^n\left(\frac{P_n}{Q_n}-\frac{A}{B}\right) < \frac{2}{Q_nD_{n+1}} < \frac{2}{Q_nQ_{n+1}}$$

or

$$0 < (-1)^n(BP_n - AQ_n) < 2B/Q_{n+1}. \tag{18}$$

It is easily shown that the inequality (18) cannot be satisfied by integers A, B with $B<Q_n$. For then the right member is less than 2, and the integer in the middle member could only be 1. By (11) the latter condition is satisfied by taking $A=P_{n-1}$, $B=Q_{n-1}$, and this is the only choice for which $0<B<Q_n$, as we shall see in §4 below. However, with $B=Q_{n-1}$, we are led from (18) to the inequality $1<2Q_{n-1}/Q_{n+1}$, which is impossible by (7).

4. Continued fractions for rational numbers. If at some stage in (1) we have $r_n=0$, then x reduces by (3) to the rational number P_n/Q_n. Conversely, every rational number is represented by a terminating continued fraction. The next to the last convergent is important in solving the linear diophantine equation (19).

Given two integers P_n and Q_n without common factor, to find all pairs of integers u and v which satisfy the equation

$$P_nu - Q_nv = 1. \tag{19}$$

The solution is given by

$$\begin{aligned} v &= (-1)^nP_{n-1} + NP_n \\ u &= (-1)^nQ_{n-1} + NQ_n \end{aligned} \qquad N \text{ any integer,} \tag{20}$$

where P_{n-1} and Q_{n-1} are obtained from the continued fraction expansion of P_n/Q_n.

5. General continued fractions. If the simple continued fraction (2) is replaced by the more general form

$$x = a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{\ddots \cfrac{}{a_{n-1} + \cfrac{b_n}{a_n + r_n}}}}} \tag{21}$$

in which r_n is not necessarily less than 1, and a_k and b_k are not necessarily positive integers, then it is readily shown that equations (7) become

$$P_n = a_nP_{n-1} + b_nP_{n-2}$$
$$Q_n = a_nQ_{n-1} + b_nQ_{n-2} \tag{22}$$

and the fundamental relation (10) may be written

$$M_n = \begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ b_1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ b_2 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ b_n & 0 \end{pmatrix}, \tag{23}$$

where we define $b_1 = Q_1 = P_0 = 1$, $Q_0 = 0$.

By taking determinants in (23), the difference between successive convergents to x is seen to equal $(-1)^n b_1b_2 \cdots b_n/Q_nQ_{n-1}$. Since the sum of these terms may not always converge, the question of convergence in this case is more complicated than that for the simple continued fractions, and we shall not discuss it at this time.

It may be of interest to include without proof the expansion

$$\frac{x}{\tan^{-1} x} = 1 + \cfrac{x^2}{3 + \cfrac{(2x)^2}{5 + \cfrac{(3x)^2}{7 + \cdots}}} \tag{24}$$

which is typical of many continued fraction expansions for analytic functions obtained from hypergeometric series.

Setting $x = 1$ in (24) we see that the value of π can be expressed as a continued fraction in which the coefficients are given by a simple law

$$\frac{4}{\pi} = 1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cdots}}}. \tag{25}$$

This expansion is of interest because of its regularity, but it does not enjoy the rapidity of convergence which characterizes the simple continued fractions. Successive convergents in (25) are given by

$$\begin{pmatrix} P_{n+1} & P_n \\ Q_{n+1} & Q_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ 9 & 0 \end{pmatrix}\begin{pmatrix} 9 & 1 \\ 16 & 0 \end{pmatrix} \cdots \begin{pmatrix} 2n+1 & 1 \\ n^2 & 0 \end{pmatrix}. \tag{26}$$

We may derive from (26) an alternative expansion for the convergents to $4/\pi$, in which the partial numerators b_i are all 1 but the partial denominators are not all integers. This is as follows:

$$(27)\quad \begin{pmatrix} P_{n+1} & P_n \\ Q_{n+1} & Q_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 5/4 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 28/9 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 81/64 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}$$

where

$$a_{2n} = (4n-1)\left(\frac{2\cdot 4\cdots(2n-2)}{3\cdot 5\cdots(2n-1)}\right)^2,$$

$$a_{2n+1} = (4n+1)\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\right)^2$$

$$(28)\quad \lim_{n=\infty} a_{2n} = \pi$$

$$\lim a_{2n+1} = \frac{4}{\pi}.$$

Since each matrix in (27) has the determinant (-1), formulas (12) and (13) are valid in this case. The limits in (28) are easily obtained from the Wallis product formula for $\pi/2$. A close estimate for the error of P_n/Q_n is $(\sqrt{2}-1)^{2n-1}$.

DERIVATION OF THE TANGENT HALF-ANGLE FORMULA

F. E. Wood, University of Oregon

The following derivation of the formula for $\tan \theta/2$ appears to be an improvement over standard derivations, for it gives the result directly without a complicated discussion of the appropriate algebraic sign.

From the equation

$$\sin\frac{\theta}{2} = \sin\left(\theta - \frac{\theta}{2}\right) = \sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}$$

one obtains

$$(1+\cos\theta)\sin\frac{\theta}{2} = \sin\theta\cos\frac{\theta}{2}.$$

Consequently:

$$\tan\frac{\theta}{2} = \frac{\sin\theta/2}{\cos\theta/2} = \frac{\sin\theta}{1+\cos\theta}.$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 851. *Proposed by Joseph Rosenbaum, Hartford, Connecticut*

The area of a quadrilateral which has both a circumcircle and an incircle is equal to the square root of the product of its sides.

E 852. *Proposed by Roy Dubisch, Fresno State College*

A theorem due to Lamé states that the number of divisions D in the Euclidean Algorithm required to find the g.c.d. of two numbers a and b $(a>b)$ is never greater than $5p$, where p is the number of digits in b. While this result has been strengthened for special pairs a, b, show, by a counter-example with b as small as possible and corresponding a as small as possible, that the statement $D<5p$ for all a, b is false.

E 853. *Proposed by C. S. Ogilvy, Trinity College*

If $y_1=x$, $y_2=x^{y_1}$, $\cdots$, $y_n=x^{y_{n-1}}$, what is the maximum x for which $\lim_{n\to\infty} y_n$ exists, and what is this limit?

E 854. *Proposed by Jerome C. R. Li, Oregon State College*

Show that $\pi=\sum_{n=0}^{\infty}(n!)^2 2^{n+1}/(2n+1)!$.

E 855. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Planes through the orthocenter of an orthocentric tetrahedron perpendicular to four concurrent cevians cut the spheres described on these cevians in four cospherical circles.

SOLUTIONS

A Combinatorial Identity

E 799 [1948, 502]. Correction to solution II [1948, 503–504].

1. On p. 504, first line and also eq. (3);

$$\text{for } \left(\frac{x-1}{2}\right)^r \text{ read } \left(\frac{x-1}{2}\right)^{n-r}.$$

2. Replace eq. (4) by

$$\sum_{r=0}^{n}\binom{n}{r}\left[\left(\frac{x+1}{2}\right)^{n-r}\left(\frac{x-1}{2}\right)^{r}\binom{n}{r}-\binom{2n-r}{n}\left(\frac{x-1}{2}\right)^{n-r}\right]=0.$$

These changes do not affect the particular application made.

Battle of Digits

E 816 [1948, 317]. *Proposed by P. L. Chessin, New York, N. Y.*

A announces a two digit number from 01 to 99. *B* reverses the digits of this number and adds to it the sum of its digits and then announces *his* result. *A* continues in the same pattern. All numbers are reduced modulo 100, so that only two digit numbers are announced. What choices has *A* for the initial number in order to insure that *B* will at some time announce 00?

Solution by Eleanor Rankin, University of North Carolina. If *A* calls 00, *B* must answer 00. For either *A* or *B* to obtain 00 he must receive a number $10a+b$ such that $11b+2a=100$. This linear Diophantine equation, subject to the restriction $0 \leqq a,b \leqq 9$, has the unique solution $a=6$, $b=8$. Continuing backwards we obtain as possible initial announcements for *A* the numbers of the unique chain 68, 16, 80, 76, 56, 64. There are no other possible numbers since the equation $11b+2a=64$ has no solution in admissible a and b.

If *B* is to be the first one to call 00, then *A*'s initial announcement is limited to the three numbers 68, 80, 56.

Also solved by Murray Barbour, Walter Breen, Paul Brock, R. C. Buck, W. E. Byrne, Monte Dernham, Roy Dubisch, J. C. Eaves, W. J. Graves, Martha Grogan, B. A. Hausmann, O. H. Hoke, B. R. Leeds, H. R. Leifer, Roger Lessard, Nathaniel Macon, D. W. Matlack, J. R. McDonough, W. D. Peeples, Jr., T. L. Reynolds, F. W. Saunders, C. W. Trigg, and the proposer.

Trigg pointed out that the initial announcement of an odd number will insure that all subsequent calls will be odd. He then gave the following groupings into which the even numbers fall in direct order of announcement. An apostrophe (') indicates the beginning of a repetitive group, and an asterisk (*) precedes a number which has been reduced modulo 100.

64, 56, 76, 80, 16, 68, *00
'54, 54
'36, 72, 36
'18, 90, 18
'12, 24, 48, 96, 84, 60, 12
20, '04, 44, 52, 32, 28, 92, 40, 08, 88, *04
86, 82, 38, 94, 62, 34, 50, 10, '02, 22, 26, 70, 14, 46, 74, 58, 98, *06, 66, 78, *02
42, 30, 06, *etc.*, as in line above.

Dernham indicated a simple algorithm for finding the chain leading to a given even terminal number. With the terminal number as the first dividend and 22 as the divisor, take twice the quotient to form the terminal digit of the second dividend, and half the remainder to form its penultimate digit. Continue this process as long as only single digits are obtained; then stop. The several dividends give the desired chain. The divisor is always 22. For example, here is all we need write to compute the chain leading to 00:

Div	*Q*	*R*	*R/2*	*2Q*
100	4	12	6	8
68	3	2	1	6
16	0	16	8	0
80	3	14	7	6
76	3	10	5	6
56	2	12	6	4
64	2	20	10	4

Since 10 is not a single-digit integer, it is inadmissible, and we have to stop. The desired chain is 64, 56, 76, 80, 16, 68, 00.

Byrne stated that the analogous problem involving three digits has for solution only the two numbers 779 and 367.

Flexilinear Curves

E 817 [1948, 317]. *Proposed by E. V. Hofler, Colgate University*

If the graph for a polynomial of the fourth degree has two real points of inflection, then the secant through these two points and the curve will bound three distinct areas. Show that two of these areas are equal and the largest area is equal to the sum of the other two.

I. *Solution by W. E. Byrne, Virginia Military Institute.* By a rigid displacement of the coordinate axes we may take the inflection points as $(-a, 0)$ and $(a, 0)$, $a>0$. We then have

$$y'' = 12k(x^2 - a^2), \qquad k \neq 0,$$

whence we find

$$y = k(x^4 - 6a^2x^2 + 5a^4) = k(x^2 - a^2)(x^2 - 5a^2).$$

Now

$$\int_{-a\sqrt{5}}^{-a} y\,dx = \int_{a}^{a\sqrt{5}} y\,dx = -16ka^5/5, \qquad \int_{-a}^{a} y\,dx = 32ka^5/5.$$

Since $|32ka^5/5| = 2|-16ka^5/5|$, the theorem is proved.

II. *Solution by Alan Wayne, Flushing, N. Y.* The proposition to be proved is a special instance of the following theorem:

If the points of inflection of a polynomial curve are all real and collinear, then the sum of the signed areas bounded by the curve and the line of collinearity is zero. Also, if these areas be denoted from left to right by $A_1, A_2, \cdots, A_{n-1}$, *then* $A_i = (-1)^n A_{n-i}$, $(i=1, \cdots, n-1)$.

Let the polynomial curve, with points of inflection on the line

$$y = mx + b,$$

be designated by

$$(1) \qquad y = \Sigma c_r x^r, \qquad n > 2, \qquad c_n = 1,$$

the summation extending over $r=0, 1, 2, \cdots, n$. Then

$$y = \Sigma b_r x^r = (\Sigma c_r x^r) - mx - b \tag{2}$$

represents a curve whose points of inflection all lie on the x-axis, and have the same abscissas as those of (1). To each of the areas A_i bounded by the curve (1) and the line $y=mx+b$, there corresponds in a one to one manner an equal area B_i bounded by the curve (2) and the x-axis.

Without loss of generality we may denote the abscissas of the x-intercepts of curve (2), from left to right, by $-a, a_2, a_3, \cdots, a_{n-1}$, and a. Then an alternative form of the equation of the curve (2) is

$$y = (x^2 - a^2)(x - a_2)(x - a_3) \cdots (x - a_{n-1}). \tag{3}$$

For the second derivative, because of the collinearity of the points of inflection of (2), we must have

$$y'' = n(n-1)(x - a_2)(x - a_3) \cdots (x - a_{n-1}). \tag{4}$$

Hence

$$n(n-1)y = (x^2 - a^2)y''. \tag{5}$$

Obtaining the expressions for y and y'' from (2), and substituting them in (5), we find that

$$n(n-1)\Sigma b_r x^r = \Sigma r(r-1)b_r x^r - a^2\Sigma r(r-1)b_r x^{r-2}. \tag{6}$$

Equating coefficients of like powers of x, we obtain

$$n(n-1)b_r = r(r-1)b_r - a^2(r+2)(r+1)b_{r+2}, \tag{7}$$

in which $b_r=0$ if $r>n$. Then, since $b_{n-k}=0$ for odd k, the curve (2) is symmetric. For odd n, the point of symmetry is the origin; for even n, the line of symmetry is the y-axis.

Let A be the sum of the signed areas bounded by the curve (2) and the x-axis, and to the right of the y-axis. Then

$$A = \Sigma b_r a^{r+1}/(r+1).$$

Integrating both sides of (6) from zero to a, we find

$$\begin{aligned} n(n-1)A &= \Sigma r(r-1)b_r a^{r+1}/(r+1) - a^2\Sigma r(r-1)b_r a^{r-1}/(r-1) \\ &= -2\Sigma r b_r a^{r+1}/(r+1) = 2\Sigma[1-(r+1)]b_r a^{r+1}/(r+1) \\ &= 2\Sigma b_r a^{r+1}/(r+1) - 2a\Sigma b_r a^r = 2A. \end{aligned}$$

Hence, since $n>2$, $A=0$.

From the symmetry of the curve (2), the sum of the areas B_i is zero, and $B_i=(-1)^n B_{n-i}$. But $A_i=B_i$, and the theorem is established.

From the recursion relation (7), noting that $b_n=1$, we may determine the coefficients b_r of the equation of the curve (2):

$$b_{n-2k} = \frac{a^{2k}n(n-1)\cdots(n-2k+1)}{(-2)^k k!(2n-3)(2n-5)\cdots(2n-2k-1)} \tag{8}$$

for integers k such that $n \geqq 2k > 0$. These curves will serve as interesting examples in courses in college algebra and calculus.

We might call curves, whose flexes are real and collinear, *flexilinear curves*. The curve $y = \sin x$ is an instance of a transcendental flexilinear curve. The Bernouilli polynomials of the third, fourth, and fifth degrees have graphs which are flexilinear polynomial curves.

Also solved by Paul Brock, F. E. Cothran, Ragnar Dybvik, Walter Fleming B. A. Hausmann, Roger Lessard, G. M. Merriman, Norman Miller, Leo Moser, S. T. Parker, R. W. Rector, J. H. Simester, R. P. Stephens, Sieh Su, C. W. Trigg, and the proposer.

Editorial Note. Not only is the proposed problem a special case of Wayne's general theorem, but so also is the well known fact that the inflection point of a cubic polynomial curve is a center of symmetry for the curve.

It is interesting that reality and collinearity of the inflection points implies symmetry for the associated curve (2)—symmetry in an axis if n is even, and symmetry in a point if n is odd. This fact can be used to give some further geometrical properties of flexilinear polynomial curves. Thus, suppose n even, and let $P_1, \cdots, P_n$ be the intersections, from left to right, of the curve with the line L of collinearity of its inflection points. Let V denote the axis of symmetry of the associated curve (2). Then we have: (A) V bisects the segment joining P_i and P_{n-i+1}. (B) Tangents to the curve at P_i and P_{n-i+1} intersect on V. (C) The bitangent of the curve touching the curve between P_i and P_{i+1} and between P_{n-i} and P_{n-i+1} is parallel to L. (D) The tangent to the curve at the point where it cuts V is parallel to L.

Other general theorems can be given for the case where n is odd.

Trigg gave twenty-five theorems and seven corollaries concerning the quartic polynomial curve with real inflections. For some properties of this curve see J. S. Frame, *Tangent triangles to a biquadratic curve*, this MONTHLY, Oct. 1944, pp. 445–452.

Four Fractions

E 818 [1948, 317]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find four different fractions, each of the form $m/(m+1)$, such that their sum is an integer. (For example: $1/2+2/3+6/7+41/42=3=1/2+2/3+8/9+17/18$.)

Solution by C. W. Trigg, Los Angeles City College. For any one of the fractions, f, we have $1/2 \leqq f < 1$. Therefore, $2 < \Sigma f < 4$, so $\Sigma f = 3$. Let the numerators of the fractions, arranged in ascending order of magnitude, be m, n, p, q, respectively. Now $2/3+3/4+4/5+5/6=183/60>3$, so $m=1$. Also, $1/2+5/6+6/7+7/8=515/168>3$, so $n<5$. The sum of the fractions may be written as

$$1/2 + n/(n+1) + p/(p+1) + q/(q+1) = 3,$$

which may be manipulated into the form

$$q = [(n+3)p + (3n+5)]/[(n-1)p - (n+3)].$$

When $n=2$, $q=5+36/(p-5)$, so $(p, q)=(6, 41)$, $(7, 23)$, $(8, 17)$, or $(9, 14)$. When $n=3$, $q=3+16/(p-3)$, so $(p, q)=(4, 19)$ or $(5, 11)$. When $n=4$, $(p, q)=(3, 19)$. Hence the four solutions additional to those given in the proposal are: $1/2+2/3+7/8+23/24=1/2+2/3+9/10+14/15=1/2+3/4+4/5+19/20=1/2+3/4+5/6+11/12=3$.

Also solved by Murray Barbour, Paul Brock, W. E. Byrne, Roger Lessard, S. T. Parker, P. A. Piza, and Kirk Stewart, and partially solved by W. H. Breen, B. A. Hausmann, Leo Moser, and C. S. Ogilvy.

Several solvers pointed out that this problem is equivalent to finding four distinct integers in the list 2, 3, 4, $\cdots$ the sum of whose reciprocals is unity.

Euler's Constant

E 819 [1948, 317]. *Proposed by H. F. Sandham, Trinity College, Ireland*

If

$$S_n = 1/1 + 1/2 + \cdots + 1/n,$$

prove that

$$\gamma < S_p + S_q - S_{pq} \leqq 1,$$

where γ is Euler's constant.

I. *Solution by S. T. Parker, Kansas State College.* We have

$$\begin{aligned} S_p + S_q - S_{pq} &= (S_p + S_q - S_p) - (S_{2p} - S_p) - (S_{3p} - S_{2p}) - \cdots \\ &\quad - (S_{qp} - S_{(q-1)p}) \\ &\leqq S_q - p(1/2p) - p(1/3p) - \cdots - p(1/qp) = 1. \end{aligned}$$

Also we have

$$\gamma < S_{pq} - \ln pq \leqq S_p - \ln p.$$

Therefore

$$0 \leqq S_p - S_{pq} - \ln p + \ln pq.$$

Add to this last the relation

$$\gamma < S_q - \ln q,$$

and we obtain

$$\gamma < S_p + S_q - S_{pq}.$$

Therefore

$$\gamma < S_p + S_q - S_{pq} \leqq 1.$$

II. *Solution by the Proposer.* If we set

$$(p, q) = S_p + S_q - S_{pq}$$

then

$$\begin{aligned}(p, q) - (p - 1, q) &= 1/p - 1/(pq - q + 1) - 1/(pq - q + 2) - \cdots - 1/pq \\ &< 1/p - q(1/pq) = 0.\end{aligned}$$

Thus (p, q) is a decreasing function of p and therefore also of q. The cases $p \to \infty$, $q \to \infty$ and $p = 1$, $q = 1$ are the two sides of the inequality.

Also solved by Michael Aissen, W. E. Byrne, William Gustin, M. S. Klamkin, Roger Lessard, Norman Miller, Leo Moser, Margaret Olmsted, and J. H. Simester.

Factorial as the Difference of Squares

E 822 [1948, 365]. *Proposed by W. R. Ransom, Tufts College*

Every factorial that can be expressed as the difference of two squares can be so expressed in two different ways.

Solution by N. G. Gunderson, University of Rochester. It is clear from the expressions

$$N! = a^2 - b^2 = (a + b)(a - b)$$

and

$$N! = AB = \{(A + B)/2\}^2 - \{(A - B)/2\}^2$$

that the number of different ways of expressing $N!$ as the difference of two squares is the number of different ways of factoring $N!$ into two factors both even or both odd. But since $N!$ is even for $N > 1$, these factors must be even. Therefore N must be greater than 3 if there are to be such factors.

To express $N!$ as $2^i 3^j 5^k \cdots$ we put

$$i = [N/2] + [N/2^2] + \cdots$$

and similarly for j, k, $\cdots$. The number of divisors, and hence the number of factorizations into two factors, of $3^j 5^k \cdots$ is $(j+1)(k+1) \cdots$. Corresponding to each of these factorizations there are $(i-1)$ factorizations of $2^i 3^j 5^k \cdots$ in which each factor is even. But, since $N!$ is never a square for $N > 1$, each such factorization appears twice, and so the number of different ways of factoring $N!$ for $N \geqq 4$ into even factors is

$$\tfrac{1}{2}(i - 1)(j + 1)(k + 1) \cdots .$$

This number is at least 2, since $i \geqq 3$, $j \geqq 1$ for $N \geqq 4$.

Thus there are two different representations of 4! as the difference of two squares, four for 5!, nine for 6!, *etc.*

Also solved by Michael Aissen, Murray Barbour, Barney Bissinger, Monte Dernham, E. J. Finan, M. S. Klamkin, H. L. Lee, Roger Lessard, Eric Michalup, Martin Milgram, Leo Moser, C. S. Ogilvy, Margaret Olmsted, R. P. Stephens, C. W. Trigg, and the proposer.

As a companion problem Moser proposed to prove that, for $N>2$, $N!$ cannot be expressed as the sum of two squares. This, he stated, follows immediately from two known theorems: (1) n is of the form x^2+y^2 if and only if $n=n_1^2n_2$, where n_2 has no prime factor of the form $4r+3$. (Hardy and Wright, p. 297.) (2) If $x>7/2$ there is at least one prime of the form $4r+3$ satisfying $x<p\leqq 2x$. (R. Breusch "Für Verallgemeinerung des Bertrandschen Postulates dass zwischen x und $2x$ stets primzahlen liegen," Math. Zeit., vol. 34 (1932), pp. 505–526, and P. Erdös "Über die Primzahlen gewisser arithmetischer Reihen," Math. Zeit., vol. 39 (1935), pp. 473–491.)

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4331. *Proposed by V. F. Ivanoff, San Francisco*

Let the equation, $x^n+p_1x^{n-1}+\cdots+p_n=0$, possess a root x_1 whose modulus exceeds that of every other root of the equation. Prove that

$$\lim \frac{\Delta_k}{\Delta_{k-1}} = -x_1,$$

where Δ_k is the k-rowed determinant $(k>n)$

$$\begin{vmatrix}
p_1 & p_2 & \cdot & \cdot & p_n & 0 & \cdot & \cdot & 0 & 0 \\
1 & p_1 & p_2 & \cdot & \cdot & p_n & \cdot & \cdot & 0 & 0 \\
0 & 1 & p_1 & \cdot & \cdot & \cdot & p_n & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 1 & p_1 & \cdot & \cdot & \cdot & p_{n-1} & np_n \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1 & p_1 & 2p_2 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & p_1
\end{vmatrix}.$$

4332. *Proposed by Paul Erdös, Syracuse University*

Let $a_1<a_2< \cdots$ be an infinite sequence of integers. Prove that from the sequence a_i+a_j, $i=1, 2, \cdots, j=1, 2, \cdots$, one can always select an infinite subsequence such that no element divides another.

4333. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In every system of numeration whose base B has the form am^p+1, each of the p-digit numbers

$$(am^{p-1}+1)^p,\ (2am^{p-1}+1)^p,\ (3am^{p-1}+1)^p, \cdots, [(m-1)am^{p-1}+1]^p,$$

gives rise to an infinite sequence of p-th powers. The law of formation is very simple and is sufficiently clear from the following example:

If $B=97=6\cdot 4^2+1=12\cdot 2^3+1=6\cdot 2^4+1=3\cdot 2^5+1$, then we have

$$\overline{25}^2 = 6\ \overline{43},\ \overline{24}\ \overline{25}^2 = 6\ 6\ \overline{42}\ \overline{43},\ \overline{24}\ \overline{24}\ \overline{25}^2 = 6\ 6\ 6\ \overline{42}\ \overline{42}\ \overline{43}, \cdots$$

$$\overline{49}^2 = \overline{24}\ \overline{73},\ \overline{48}\ \overline{49}^2 = \overline{24}\ \overline{24}\ \overline{72}\ \overline{73},$$

$$\overline{48}\ \overline{48}\ \overline{49}^2 = \overline{24}\ \overline{24}\ \overline{24}\ \overline{72}\ \overline{72}\ \overline{73}, \cdots$$

$$\overline{73}^2 = \overline{54}\ \overline{91},\ \overline{72}\ \overline{73}^2 = \overline{54}\ \overline{54}\ \overline{90}\ \overline{91},$$

$$\overline{72}\ \overline{72}\ \overline{73}^2 = \overline{54}\ \overline{54}\ \overline{54}\ \overline{90}\ \overline{90}\ \overline{91}, \cdots$$

$$\overline{49}^3 = \overline{12}\ \overline{48}\ \overline{85},\ \overline{48}\ \overline{49}^3 = \overline{12}\ \overline{12}\ \overline{48}\ \overline{48}\ \overline{84}\ \overline{85},$$

$$\overline{48}\ \overline{48}\ \overline{49}^3 = \overline{12}\ \overline{12}\ \overline{12}\ \overline{48}\ \overline{48}\ \overline{48}\ \overline{84}\ \overline{84}\ \overline{85}, \cdots$$

$$\overline{49}^4 = 6\ \overline{30}\ \overline{66}\ \overline{91},\ \overline{48}\ \overline{49}^4 = 6\ 6\ \overline{30}\ \overline{30}\ \overline{66}\ \overline{66}\ \overline{90}\ \overline{91},$$

$$\overline{48}\ \overline{48}\ \overline{49}^4 = 6\ 6\ 6\ \overline{30}\ \overline{30}\ \overline{30}\ \overline{66}\ \overline{66}\ \overline{66}\ \overline{90}\ \overline{90}\ \overline{91}, \cdots$$

$$\overline{49}^5 = 3\ \overline{18}\ \overline{48}\ \overline{78}\ \overline{94},\ \overline{48}\ \overline{49}^5 = 3\ 3\ \overline{18}\ \overline{18}\ \overline{48}\ \overline{48}\ \overline{78}\ \overline{78}\ \overline{93}\ \overline{94},$$

$$\overline{48}\ \overline{48}\ \overline{49}^5 = 3\ 3\ 3\ \overline{18}\ \overline{18}\ \overline{18}\ \overline{48}\ \overline{48}\ \overline{48}\ \overline{78}\ \overline{78}\ \overline{78}\ \overline{93}\ \overline{93}\ \overline{94}, \cdots .$$

Establish the general rule.

4334. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that the feet of the six perpendiculars from the Bennett point on the sides of a complete quadrangle lie on a conic.

4335. *Proposed by D. D. Wall, Cambridge, Massachusetts*

If n is a prime greater than 3, show that

$$(1) \qquad \prod_{x=1}^{n}\left(1+2\cos\frac{2\pi x}{n}\right)=3,$$

$$(2) \qquad \sum_{x=1}^{n}\sec\frac{2\pi x}{n}=(-1)^{(n-1)/2}n.$$

The restrictions on n can be lessened somewhat in each case.

SOLUTIONS

Isogonic and Isodynamic Centers

4196 [1946, 160]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1, B_1, C_1 be vertices of equilateral triangles constructed exteriorly, or interiorly, on the sides BC, CA, AB of a triangle, and let A_2, B_2, C_2 be the intersections $(BC_1, CB_1)(CA_1, AC_1)$, (AB_1, BA_1). Then, V and W being the first (or second) isogonic and isodynamic centers of ABC, show that $1/VW=1/A_1A_2+1/B_1B_2+1/C_1C_2$.

I. *Solution by O. J. Ramler, Catholic University.* From a figure one can readily see that triangles A_2BC and AB_2C are directly similar, and thus show that triangles CA_1A_2 and CB_2B_1 are also directly similar. Therefore A_1A_2 and B_1B_2 are parallel. In this manner we may show that A_1A_2, B_1B_2, C_1C_2 are parallel.

Since A_1, A_2 and V, W are pairs of isogonal conjugate points for triangle ABC, it follows that A, W, A_2 are collinear. Let AA_1 and AA_2 cut BC in V_1 and W_1 respectively. Then the points A, V, V_1, A_1 on lines AA_1 are in one-to-one isogonal correspondence with the points W_1, W, A, A_2 on line AA_2. The two sets of points are therefore projectively related, and their axis of homology is the line BC. Hence VA_2 and WA_1 intersect on BC. It now follows that triangles B_1VA_2 and B_2WA_1 are coaxal with BC as axis. These triangles are then also copolar, and VW is parallel to A_1A_2 and B_1B_2.

As a consequence of the above we have

$$VW/A_1A_2 = AV/AA_1, \qquad VW/B_1B_2 = BV/BB_1, \qquad VW/C_1C_2 = CV/CC_1.$$

But, since V is an isogonic center of triangle ABC,

$$VA + VB + VC = AA_1 = BB_1 = CC_1. \tag{1}$$

Therefore

$$1/VW = 1/A_1A_2 + 1/B_1B_2 + 1/C_1C_2.$$

II. *Solution by the Proposer.* We shall establish the more general

THEOREM. Given a triangle ABC and an arbitrary point V in its plane, let the lines AV, BV, CV cut the circles BVC, CVA, AVB again in the points A_1, B_1, C_1 and let W, A_2, B_2, C_2 denote the isogonal conjugates, for triangle ABC, of V, A_1, B_1, C_1. Then

$$1/VW = 1/A_1A_2 + 1/B_1B_2 + 1/C_1C_2.$$

Let Γ, Γ_a, Γ_b, Γ_c be the conics inscribed in triangle ABC and having V and W, A_1 and A_2, B_1, and B_2, C_1 and C_2 as foci. Then conic Γ is the envelope of segments PQ subtending at V an angle equal to $180°-BVC$, P and Q being on AB and AC. But conic Γ_a is the envelope of segments P_1Q_1 subtending at A_1 an angle equal to $180°-BVC$. To each tangent PQ of Γ corresponds a parallel tangent P_1Q_1 of Γ_a such that VP, VQ are parallel to A_1P_1, A_1Q_1. Conic Γ_a is thus the transform of Γ under the homothety $(A, AA_1/AV)$. Therefore VW and A_1A_2 are

parallel and $VW/A_1A_2 = AV/AA_1$. In this manner, by similarly treating the conics Γ_b and Γ_c, we obtain

$$VW(1/A_1A_2 + 1/B_1B_2 + 1/C_1C_2) = AV/AA_1 + BV/BB_1 + CV/CC_1.$$

But

$$AV/AA_1 + BV/BB_1 + CV/CC_1 = 1, \tag{2}$$

whence

$$1/VW = 1/A_1A_2 + 1/B_1B_2 + 1/C_1C_2.$$

We have the given problem if V is taken as one of the isogonic centers of triangles ABC.

Editorial Note. The proposer pointed out that VW, A_1A_2, B_1B_2, C_1C_2 of solution I have been shown to be parallel to the Euler line of triangle ABC by A. Boutin in the *Journal de Mathématiques Élémentaires de G. de Longchamps*, 1889.

Relation (1) above is a well known property of the isogonic centers, and may be found, for example, in art. 353 of Johnson's *Modern Geometry*. If V is not inside triangle ABC, then certain signs must accompany VA, VB, VC, and the same signs will be attached to the corresponding fractions $1/A_1A_2$, $1/B_1B_2$, $1/C_1C_2$.

Relation (2) may be established as follows; Invert the figure with respect as ρ^2 as power. Designating the inverted points by corresponding lower case letters, the inverted figure is a triangle abc with three cevians aa_1, bb_1, cc_1 concurrent in V. We have

$$AV = \rho^2/aV, \qquad BV = \rho^2/bV, \qquad CV = \rho^2/cV,$$

$$AA_1 = \rho^2/aV + \rho^2/Va_1, \qquad BB_1 = \rho^2/bV + \rho^2/Vb_1, \qquad CC_1 = \rho^2/cV + \rho^2/Vc_1.$$

Therefore

$$\begin{aligned} AV/AA_1 + BV/BB_1 + CV/CC_1 &= Va_1/aa_1 + Vb_1/bb_1 + Vc_1/cc_1 \\ &= \Delta bVc/\Delta bac + \Delta cVa/\Delta cba + \Delta aVb/\Delta acb = 1. \end{aligned}$$

A Special Tetrahedron

4201 [1946, 225]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A tetrahedron is given for which the difference of squares of opposite edges is the same for the three pairs. (1) The three medians are equal and the line joining the circumcenter to the centroid is perpendicular to one of the faces. (2) One of the altitudes passes through the symmetric of the orthocenter of the corresponding face with respect to the circumcenter of the face. (3) The sum, or the difference, of the cosines of two opposite dihedrals of the tangential tetrahedron is the same for the three pairs.

I. *Solution by the Proposer.**

(1) Let $T \equiv ABCD$ be a tetrahedron such that

$$|a^2 - a'^2| = |b^2 - b'^2| = |c^2 - c'^2| \neq 0,$$

where $BC=a$, $CA=b$, $AB=c$, $AD=a'$, $BD=b'$, $CD=c'$. We shall limit our discussion to the case where

$$a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2,$$

as the three other possibilities give rise to like conclusions.

If we denote by N, Q, S the midpoints of the edges DA, DB, DC, we find that triangles BNC, CQA, ASB are isosceles. For instance, $\overline{BN}^2=(2c^2+2b'^2-a'^2)/4$, $\overline{CN}^2=(2b^2+2c'^2-a'^2)/4$. The bimedians MN, PQ, RS are therefore perpendicular respectively to BC, CA, AB. Hence the planes drawn through N, Q, S perpendicular to BC, CA, AB (perpendicular bisectors of BC, CA, AB) intersect in a line Δ through the circumcenter O_d of triangle ABC perpendicular to the plane ABC. As the bimedians meet at the centroid G of T, Δ contains G and $GA=GB=GC$. Likewise the circumcenter O and the Monge point Ω of T are on Δ.

(2) Let G_d and H_d be the centroid and the orthocenter of triangle ABC. Let D' be the orthogonal projection of D on the plane ABC. Since G projects into O_d and $DG=3GG_d$,

$$D'O_d = 3\,O_dG_d = O_dH_d.$$

Hence D' is the symmetric of H_d with respect to O_d.

(3) Let $T_2 \equiv A_2B_2C_2D_2$ be the tangential tetrahedron of T, with the plane of the face $B_2C_2D_2$ tangent to the circumsphere (O, R) of T at A. If α is the plane angle of $A_2-B_2C_2-D_2$, and α' the plane angle of $B_2-A_2D_2-C_2$, etc., we have

$$a^2 = 2R^2(1+\cos\alpha'), \quad b^2 = 2R^2(1+\cos\beta'), \quad c^2 = 2R^2(1+\cos\gamma'),$$

$$a'^2 = 2R^2(1+\cos\alpha), \quad b'^2 = 2R^2(1+\cos\beta), \quad c'^2 = 2R^2(1+\cos\gamma),$$

and

$$\cos\alpha' - \cos\alpha = \cos\beta' - \cos\beta = \cos\gamma' - \cos\gamma,$$

provided that T is interior to T_2.

If, for example, D_2 and O are on opposite sides of the plane $A_2B_2C_2$, i.e. T is exterior to T_2, we have

$$a^2 = 2R^2(1+\cos\alpha'), \quad a'^2 = 2R^2(1-\cos\alpha), \cdots,$$

and

$$\cos\alpha' + \cos\alpha = \cos\beta' + \cos\beta = \cos\gamma' + \cos\gamma.$$

* Translated and checked by W. E. Byrne, Virginia Military Institute.

If $a^2-a'^2=b^2-b'^2=c^2-c'^2=0$, we have an isosceles tetrahedron, which is discussed on pages 94–102 of Court, *Modern Pure Solid Geometry*.

II. *Generalization by R. Blanchard, Le Havre, France.* If, using the notations of the solution by the proposer, $a^2+ka'^2=b^2+kb'^2=c^2+kc'^2=l$, k arbitrary:

(1) D' is on the line joining O_d and H_d, and $O_dD'=O_dH_d/k$.

(2) The line which joins the circumcenter O of T to the point K of the median DG_d such that $KG_d/KD=k/3$ is perpendicular to the plane ABC. The segements AA_1, BB_1, CC_1 which join A, B, C to K and are terminated at the opposite faces of T are of equal length.

(3) The centroid G of T is projected orthogonally on ABC in a point G_d' of O_dG_d such that $O_dG_d'/O_dG_d=3(k+1)/4k$.

(4) $\cos\alpha'+k\cos\alpha=\cos\beta'+k\cos\beta=\cos\gamma'+k\cos\gamma$ if T is interior to T_2.

Let us take the point D_1 on DO_d such that $O_dD_1=kO_dD$ and write $D_1A=a_1'$, $D_1B=b_1'$, $D_1C=c_1'$. We find $k\overline{DA}^2-\overline{D_1A}^2=(k-1)\overline{AO_d}^2+k\overline{DO_d}^2-\overline{D_1O_d}^2$. If R_d denotes the radius of the circumcircle of the triangle ABC, we have

$$ka'^2 - a_1'^2 = (k-1)R_d^2 - k(k-1)\overline{DO_d}^2.$$

Hence, if

$$l - (k-1)R_d^2 + k(k-1)\overline{DO_d}^2 > 0,$$

$$a^2 + a_1'^2 = l - (k-1)R_d^2 + k(k-1)\overline{DO_d}^2 = b^2 + b_1'^2 = c^2 + c_1'^2,$$

which shows that the tetrahedron $ABCD_1$ is orthocentric.

To construct a tetrahedron T satisfying the statement of the generalized problem, it is sufficient to start with an orthocentric tetrahedron $ABCD_1$, to take a point D on O_dD_1 such that $O_dD=O_dD_1/k$.

(1) D' is on O_dH_d and $O_dD'/O_dH_d=O_dD/O_dD_1=1/k$.

(2) Let L be the point where the perpendicular at G_d to the face ABC meets DD_1. Since $KG_d/KD=k/3$ and $O_dL/O_dD=O_dD_1/3O_dD=k/3$, KO_d and G_dL are parallel. Hence O_dK is perpendicular to the face ABC and passes through O. Furthermore, as K is the centroid of A, B, C, D with the weights $1, 1, 1, -k$,

$$AA_1/AK = BB_1/BK = CC_1/CK = (3-k)/(2-k)$$

and since $AK=BK=CK$, it follows that $AA_1=BB_1=CC_1$.

(3) We have

$$O_dG_d' = O_dG_d + G_dG_d' = O_dG_d + G_dD'/4.$$

But

$$G_dD' = O_dD' - O_dG_d = O_dG_d(3-k)/k.$$

Hence

$$O_dG_d' = 3(n+1)O_dG_d/4k.$$

(4) If T is interior to T_2,

$$a^2 + ka'^2 = 2R^2[1 + \cos\alpha' + k(1 + \cos\alpha)],$$

$$\cos\alpha' + k\cos\alpha = \cos\beta' + k\cos\beta = \cos\gamma' + k\cos\gamma = (l/2R^2) - (k+1).$$

If $k=-1$, the results are:

(1) $O_dD'/O_dH_d = -1$, D' is the symmetric of H_d with respect to O_d.

(2) $KG_d/KD = -1/3$, K coincides with G. OG is perpendicular to the plane ABC, and the medians AA_1, BB_1, CC_1 are equal.

(3) $O_dG_d' = 0$, which shows again that OG is perpendicular to the face ABC. We find the question proposed by V. Thébault as a special case of our generalization.

If $k=3$, D' coincides with G_d and conversely if $D' \equiv G_d$, $k=3$. The necessary and sufficient condition that the orthogonal projection of D on ABC be the centroid of ABC is that $a^2+3a'^2 = b^2+3b'^2 = c^2+3c'^2$.

Likewise, the necessary and sufficient condition that the orthogonal projection of D on ABC be the center of the nine-point circle of ABC is that

$$a^2 + 2a'^2 = b^2 + 2b'^2 = c^2 + 2c'^2.$$

Isogonal Conjugate Points of a Tetrahedron

4208 [1947, 50]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given an orthocentric tetrahedron. If two isogonal conjugate points are also conjugate with respect to the circumsphere, their pedal sphere is orthogonal to the sphere, belonging to the linear net determined by the circumscribed and conjugate spheres, and whose center is the complementary point of the orthocenter with respect to the tetrahedron.

*Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France.** We choose rectangular axes with the origin at the orthocenter H of the orthocentric tetrahedron $T \equiv ABCD$, and with the circumcenter O of T on the x-axis at $(d, 0, 0)$. The conjugate sphere (H) of T is given by

$$(H) \equiv x^2 + y^2 + z^2 - \rho^2 = 0$$

with $d^2 = R^2 + 3\rho^2$.† The circumsphere (O) of T has the equation

$$(O) \equiv (x-d)^2 + y^2 + z^2 - R^2 = 0,$$

or

$$(O) \equiv x^2 + y^2 + z^2 - 2dx + 3\rho^2 = 0.$$

Let $(\Gamma) \equiv b^2(u^2+v^2+w^2) - (ux_1+vy_1+wz_1+s)(ux_2+vy_2+wz_2+s) = 0$ be the tangential equation of a quadric of revolution with the given isogonal conjugate points $F_1(x_1, y_1, z_1)$, $F_2(x_2, y_2, z_2)$ as foci. The condition that (Γ) be inscribed in T is:

* Translated by W. E. Byrne, Virginia Military Institute.

† Court, Modern Pure Solid Geometry, p. 265.

$$3J^2 - x_1x_2 - y_1y_2 - z_1z_2 + \rho^2 = 0. \tag{1}$$

Since F_1, F_2 are conjugate with respect to (O), we have

$$x_1x_2 + y_1y_2 + z_1z_2 - d(x_1 + x_2) + 3\rho^2 = 0. \tag{2}$$

The pedal sphere (S) of F_1 and F_2 is:

$$\left(x - \frac{x_1 + x_2}{2}\right)^2 + \left(y - \frac{y_1 + y_2}{2}\right)^2 + \left(z - \frac{z_1 + z_2}{2}\right)^2 = b^2 + \left(\frac{x_1 - x_2}{2}\right)^2 + \left(\frac{y_1 - y_2}{2}\right)^2 + \left(\frac{z_1 - z_2}{2}\right)^2,$$

or

$$(S) \equiv x^2 + y^2 + z^2 - (x_1 + x_2)x - (y_1 + y_2)y - (z_1 + z_2)z + x_1x_2 + y_1y_2 + z_1z_2 - b^2 = 0.$$

The condition that the sphere

$$\lambda(H) + (O) = 0 \tag{3}$$

of the coaxal pencil determined by (H) and (O), be orthogonal to (S) is that $\lambda = \frac{1}{2}$. For $\lambda = \frac{1}{2}$, equation (3) reduces to

$$x^2 + y^2 + z^2 - 4dx/3 + 5\rho^2/3 = 0, \tag{4}$$

or

$$(x - 2d/3)^2 + y^2 + z^2 = 4d^2/9 - 5\rho^2/3 = R^2/9 + (a^2 + a'^2)/12,$$

where $a = BC$, $a' = AD$.* The center $\Omega(2d/3, 0, 0)$ of the sphere (4) is the complementary point of H with respect to T since

$$\overrightarrow{G\Omega} = -\overrightarrow{GH}/3.$$

The Rebounding Projectile

4249 [1947, 286]. *Proposed by W. B. Campbell, Philadelphia Textile Institute*

A body is projected from a point O in a plane making an angle A with the horizontal, the direction of projection being in a vertical plane containing a line of greatest slope of the plane, and making an angle B with the upward direction of that line. If the plane be smooth and the body perfectly elastic, derive expressions for t_n, the time consumed in the nth flight, and for x_n, the coördinate of the point of impact at the end of the nth flight. Will it ever strike O again, and will any of its flights be vertical? What is maximum x_n?

Solution by F. G. Fender, Rutgers University. Take OX up the plane and OY normal to it, and let v be the initial velocity. Then

$$y = vt \sin B - \tfrac{1}{2}gt^2 \cos A$$

* For the details see Court, loc. cit., p. 271.

for the first arch y returns to zero at

$$t = 2v \sin B/g \cos A \equiv T$$

and, at the end of the first flight

$$\dot{y} = v \sin B - \tfrac{1}{2}g \cos A(2 \cdot 2v \sin B/g \cos A) = -v \sin B.$$

Then, by the perfect elasticity, each new arch has the same initial y-component of velocity, and hence t_n the time of flight in each arch will be $\Delta t = T$ a constant. If x_n is the value of x at the end of the nth arch, x_n is also the value of x at the beginning of the $(n+1)$th arch. Similarly u_n, the velocity at the end of the nth arch, is also the velocity at the beginning of the $(n+1)$th arch. Then

$$\Delta x = x_n - x_{n-1} = u_{n-1}T - \tfrac{1}{2}gT^2 \sin A.$$

Also $u_n = u_{n-1} - gT \sin A$. Since $u_0 = v \cos B$ we have

$$u_n = u_0 - ngT \sin A = v \cos B - 2nv \sin B \tan A$$

or

$$(1) \qquad u_n = v \sin B \tan A(\cot A \cot B - 2n) = K(L - 2n), \qquad n = 0, 1, 2, \cdots,$$

where

$$K \equiv v \sin B \tan A, \qquad L \equiv \cot A \cot B.$$

Then

$$(2) \quad \Delta x = K[L - 2(n-1)]T - \tfrac{1}{2}gT^2 \sin A = KT(L - 2n + 1), \quad n = 1, 2, \cdots.$$

Furthermore we have

$$(3) \qquad x_n = \sum_{j=1}^{n} \Delta x = KT \sum_{j=1}^{n} (L + 1 - 2j) = KTn(L - n).$$

If A and B are assumed acute, then K, L, T are all positive. Then, no matter how large L may be, Δx is negative for all n greater than $\frac{1}{2}(L+1)$. Hence the maximum value of x_n occurs when n equals the smallest integer which exceeds $\frac{1}{2}(L+1)$. For greater values of n, Δx is always negative and increases indefinitely in magnitude, so that $x_n \to -\infty$ as $n \to \infty$.

If L is any odd positive integer, $2m+1$, then x_m and x_{m+1} are equal, so that the $(m+1)$th flight is vertical. Also $x_{m+k+1} = x_{m-k}$ $(0 \leqq k \leqq m)$. If L is a positive even integer, $2m$, then the mth flight ends normal to the plane, $n = m+1$ gives the maximum x_n, hence also $x_{m+k} = x_{m-k} (0 \leqq k \leqq m)$. From (3) if $x_n = x_{n'}$, n and n' integers, we must have $n = n'$ or $n - n' = L$. Thus the body never strikes twice in the same place except when L is a positive integer.

It is interesting to note, and easily shown, that the maximum abscissa, x_{max} (with positive y) is given by

$$x_{max} = \tfrac{1}{4}KTL^2,$$

which exceeds maximum x_n in all cases except when L is a positive even integer.

Also solved by N. J. Fine and the Proposer.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Solid Analytic Geometry. By J. M. H. Olmsted. New York, D. Appleton-Century Company, Inc., 1947. 13+257 pages. $4.00.

That solid analytic geometry constitutes a recognizable subject matter seems generally agreed. In one view the parts are loosely assembled from other disciplines to which they properly belong. In another, those same parts cohere in a reasonably well defined discipline. The author has produced a comprehensive text in which the latter view preponderates.

The purpose of the book, as declared in the preface, is to meet the needs of any of several courses: a brief course emphasizing planes and lines and treating only the simpler equations of second degree; a standard first course which would include the notion of rank of a matrix, coordinate transformations, and a more complete study of equations of second degree; and an extensive course enlarging the study of planes and lines and employing matrix methods in the treatment of transformations and the general equation of second degree. Basing himself squarely on coordinates, the author carries through this program. To provide the required flexibility he seeks to postpone difficulties as much as possible and to combine simplicity of presentation with authenticity.

Beginning with the properties of Euclidean space, coordinates are introduced and the necessary preliminaries disposed of. There follows a treatment of planes and lines. In the succeeding chapter facts about determinants are recalled and matrix and rank are defined. (An outline proof of the invariance of rank under elementary transformations is included.) Systems of planes, and of simultaneous linear equations generally, are dealt with on this basis. The chapter concludes with a brief account of finite dimensional real and complex space.

The major concern of the text is the study of surfaces, restricted to be algebraic. Associated with a real polynomial equation $f(x, y, z)=0$ are a graph and an algebraic surface. The algebraic surface consists of irreducible surfaces (graphs), each with a multiplicity. General properties of surfaces (and of curves of intersection) are treated first, as well as such special sorts of surfaces as cylinders and the like. A chapter is then devoted to the seventeen canonical quadric surfaces, identified by their equations. (Only real graphs are analyzed, although two surfaces are regarded as distinct if their complex graphs are distinct.) With these "standard" quadric surfaces at hand, a preliminary analysis of the general equation of second degree is undertaken. This first attack, utilizing matrices associated with the equation, proceeds without transformations and is concentrated upon such matters as centers, tangents, and rulings. The following chapter treats of translations and rotations as transformations either of coordinates

or of points. The ultimate goal is attained in proving that an equation of second degree is reducible under rigid motions (reflections included) to one and only one of the canonical seventeen.

A last and essentially independent chapter develops the elements of matrix algebra.

The author proposes a course of some thirty lessons, more or less. Much additional material suitable for an extended version or for supplementary reading is available. The text abounds with exercises ranging from routine to penetrating. Suggestions and illustrative examples are disposed at strategic places.

The plan of the book entails the avoidance of algebraic complications. For this reason, while the author does not demand it, a familiar acquaintance with parts of algebra is certainly desirable. Determinants are of frequent occurrence and their properties are used. Crucial theorems from the theory of real polynomials are quoted, with references, but their significance may escape the uninitiated. It is stated, for example, that polynomials can be decomposed into irreducible factors essentially uniquely and that a polynomial which vanishes everywhere is the zero polynomial.

Two features of the presentation should be remarked. First, the definition of equivalence for real polynomial equations in terms of their graphs is eschewed. Instead equivalence is based on these transformations of an equation: multiplication of both members by the same nonzero real number and addition to both members of the same polynomial. (By tacit agreement equal polynomials, as xy and yx, for example, may replace each other anywhere.) Second, an algebraic surface is not the graph, real or complex, of a real polynomial equation $f(x, y, z) = 0$. To each of the distinct real irreducible factors $g(x, y, z)$ of $f(x, y, z)$ corresponds an irreducible surface, the complex graph of $g(x, y, z) = 0$. The totality of these graphs, each bearing the multiplicity of $g(x, y, z)$ as a factor of $f(x, y, z)$, constitutes the algebraic surface. This choice of definitions makes for a certain amount of fussiness which largely evaporates for quadric surfaces. Throughout, the real part of a surface naturally predominates, the nonreal part making only incidental appearances.

The reviewer found no misprints, contrary to custom, but did note these places where misunderstanding is possible. A scrupulous effort to verify the assertion at the top of page 137 that ϕ must vanish under all circumstances requires actual examination of the canonical quadric surfaces. And the figure of section 11 is still misleading in spite of the author's caveat.

A. L. PUTNAM

The Outlook for Women in Mathematics and Statistics. By Mary H. Brilla. Bulletin No. 223–4 of the Women's Bureau, U. S. Department of Labor. Washington, Government Printing Office, 1948. 10+21 pages. $0.10.

This small booklet is part of a larger study of occupational opportunities for women in all scientific fields. The great demand for women with scientific train-

ing during World War II prompted the Women's Bureau to make this study. An attempt is made to tell where these women work, what kind of work they are doing, and what other young women may expect in the way of future opportunities to do work of this type.

Although written primarily for women, much of the material in the booklet is of interest to all who expect to earn a living by following some mathematical profession. Estimates are made of the number of mathematicians in teaching and in other fields. The prewar distribution of mathematicians, the average number of annual additions, wartime changes, present earnings and advancement, and the outlook for the future are other topics treated in detail. The appendix contains a bibliography which should prove useful to all who are concerned with the advisement of students of mathematics.

H. M. GEHMAN

Calculus and its Applications. By R. D. Douglass and S. D. Zeldin. New York, Prentice-Hall, Inc., 1947. 8+568 pages. $5.15.

In the preface the authors state that their purpose in writing this textbook on the Calculus was to cover the material essential for students in engineering and science in a shorter period of time than that now required in most schools and colleges. While it is true that they have omitted some topics normally included in a text of this type and have discussed other topics more briefly than is usual, the book contains ample material for a full year's course and nearly all the topics that are usually included in such a course.

The method of presentation employed throughout the book consists of short discussions or statements of purpose followed by numerous illustrative examples solved in great detail. In fact, one of the excellent features of the book is the many well chosen illustrative examples. There are more than 360 of these and nearly all are treated with unusual care and thoroughness.

An excellent treatment of differentials is given in Chap. IV followed immediately by the introduction to the indefinite integral. Integration is defined as the process of finding a function $F(x)$ which has for its differential a given expression $f(x)dx$. Moreover, answers are verified by finding their differentials so that students should not so readily adopt the annoying habit of treating dx as excess baggage.

The definite integral is postponed until Chap. VI by which time the student has had ample drill in formal integration and the applications of the indefinite integral. This chapter is clearly written and well illustrated by examples.

The first part of Chap. XIII (infinite series) is very well done. This reviewer agrees wholeheartedly with the authors that the integral test, when properly presented, is perhaps the simplest technique for the beginner to understand and should be introduced early.

Other parts of the book not specifically mentioned above contain most of the material normally found in a book of this type, including a chapter on dif-

ferential equations. In addition, there is a chapter on vectors. This final chapter (on vectors) embraces no material that could not be taught to college sophomores. The entire book is carefully written, is developed in a logical fashion, and appears to be very teachable. There are nearly 200 figures the majority of which are excellent. A list of problems is included in nearly every section and a review list is found at the end of each chapter. There are over 2700 problems in the book and the answers to all problems are given.

Naturally there are some criticisms of an adverse nature that this reviewer feels should be made. Parametric equations are not introduced until Chap. VIII, and polar equations are withheld until Chap. IX. These can profitably be used much earlier. Why the authors treat only the indeterminate form 0/0 when some of the others are so easily converted into this form is puzzling. Why a chapter on vectors is included and such standard topics as approximate integration, cylindrical and spherical coordinates, and a fuller discussion of plane curves are omitted is also hard to understand. The chapter on partial differentiation is somewhat out of balance. A lengthy and rather complete discussion of partial and total derivatives of composite functions is followed by a weak and inadequate treatment of maxima and minima.

A few errors were discovered, most of which were either trivial or mildly serious. For example, on page 33 there is a delta t where it should be delta x; on page 74, the answer to example 3 should be verified; on page 146, the lower limit of sigma is missing; and in all about 20 errors of this type were noted. There were a few of a slightly more serious nature such as stating that the circle of curvature has a radius equal to the radius of curvature with no mention of sign; or in discussing approximating by differentials the statement is made "By means of differentials, however, the calculation is shortened, and even though the resulting value *may be* approximate, the error made *is* negligible." (Italics mine.) An interchange of verbs would improve this statement. In the discussion on fluid pressure on a vertical surface, page 168, the statement is made "if delta h *is small*, the surface of each rectangular element is approximately parallel to the surface of the liquid." This obviously is false as the element remains vertical regardless of the size of delta h.

Finally a word concerning the topography. The paper is a good grade and the printing is clear excepting for fractional exponents. These are nearly always difficult to read and are frequently almost illegible. There are some bad breaks in equations such as $y=3^{1/2}$ on one line and sin x on the next line. The same size type is used for text, illustrative examples, and problems. This makes it difficult to determine where an illustrative example ends and a new discussion begins.

In summary, this reviewer feels that this is a very carefully written text with remarkably few inaccuracies most of which are of a trivial nature. Since the good points of this book far outweigh the weak points, it is a worthy addition to the present list of texts on the Calculus.

P. M. HUMMEL

NEW BOOKS RECEIVED

A Source Book in Greek Science. By M. R. Cohen and I. E. Drabkin. New York, McGraw-Hill Book Co., 1948. 22+597 pages. $9.00.

Mathematics of Finance. By P. M. Hummel and C. L. Seebeck. New York, McGraw-Hill Book Co., 1948. 11+365 pages. $4.00.

A Philosophy of Mathematics. By L. O. Kattsoff. Ames, The Iowa State College Press, 1948. 9+263 pages. $5.00.

The Kelley Statistical Tables. Revised Edition. By T. L. Kelley. Cambridge, The Harvard University Press, 1948. 9+223 pages. $5.00.

Rinehart Mathematical Tables. By H. D. Larsen. New York, Rinehart and Co., 1948. 8+264 pages. $1.50.

Differential- und Integralrechnung im Hinblick auf ihre Anwendungen. By L. Locher-Ernst. Basel, Birkhauser, 1948. 594 pages. 48 Fr.

Spherical Harmonics. Second Revised Edition. By T. M. MacRobert. New York, Dover Publications, 1948. 15+372 pages. $4.59.

Bericht über die Mathematiker-Tagung in Tübingen vom 23 bis 27 September 1946. The Mathematics Institute, Tübingen University. Tübingen, Druck von H. Laupp, Jr., 1946. 143 pages.

Leibniz. By J. T. Merz. New York, Hacker Press, 1948. 8+216 pages. Lithoprint Edition. $2.75.

Number Theory and Its History. By Oystein Ore. New York, McGraw-Hill Book Co., 1948. 10+370 pages. $4.50.

Industrial Electric Furnaces and Appliances. Vol. 2. By V. Paschkis. New York, Interscience Publishers, 1948. 14+320 pages. $8.00.

Mendeleyev. By D. Q. Posin. New York, Whittelsey House, McGraw-Hill Book Co., 1948. 12+345 pages. $4.50.

The Mathematical Solution of Engineering Problems. By M. G. Salvadori and K. S. Miller. New York, McGraw-Hill Book Co., 1948. 10+245 pages. $3.50.

Tables of Coefficients for Obtaining the First Derivatives Without Differences. (National Bureau of Standards, Applied Math. Series 2.) By H. E. Salzer. Washington, U. S. Govt. Printing Office, 1948. 20 pages. $0.15.

Mathematics Our Great Heritage. Essays on the Nature and Cultural Significance of Mathematics. By W. L. Schaaf. New York, Harper and Brothers, 1948. 11+291 pages. $3.50.

The Mathematical Basis of the Arts (Parts 1, 2, 3). By J. Schillinger. New York, Philosophical Library, 1948. 10+696 pages. $12.00.

Tables of the Bessel Functions of the First Kind of Orders Twenty-eight Through Thirty-nine. (Annals of the Computation Laboratory of Harvard University, No. 10.) Cambridge, The Harvard University Press, 1948. 694 pages. $10.00.

Lezioni de Analisi Matematica. Sixth Edition. By F. Tricomi. Padova, Cedam, 1948. 12+335 pages.

Theory of Equations. By J. V. Uspensky. New York, McGraw-Hill Book Co., 1948. 7+353 pages. $4.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITOR'S NOTE. Many of the papers presented by undergraduates at mathematics club meetings merit publication. It has been called to our attention that *The Pentagon*, the official publication of Kappa Mu Epsilon, will accept some worthy papers written by undergraduates and submitted by a responsible member of the faculty.

It is not necessary, at present, for the student to be a member of Kappa Mu Epsilon in order for his paper to be accepted. Manuscripts should be sent to the editor of *The Pentagon*, Prof. Harold D. Larsen, Albion College, Albion, Michigan.

CLUB REPORTS, 1947–48

Mathematics Club, Brown University

The following papers were presented during the thirty-fourth annual program of the *Mathematics Club* of Brown University:

Numbers, by Dr. R. P. Boas, Jr., Executive Editor of the *Mathematical Reviews*

Some geometric extremal problems, by Anthony Behr

End-point maxima and minima, by Edward Fisher

An isoperimetric problem, by Ann Patterson

The snow plow problem, by Donald Haas

On the relation between mathematics and physics, by Prof. R. B. Lindsay, Professor of Physics at Brown University

Napoleon's problem, by Beverly Woonton

The game of Nim, by Reginald Flanders

Descartes, by Ann Kernan

Mechanical integrators, by Loren Wood.

Each of the regular meetings concluded with a social and discussion period at which light refreshments were served. The Mathematics Club presented a 15-minute program over station WBRU of Brown University on the history, program and purposes of the organization.

The Committee on Program and Arrangements for the year 1947–48 consisted of: L. W. Peckham, Jr., chairman; T. C. Andrews; P. J. Bray; K. M. Crowe; P. C. Curtis, Jr.; E. T. Mansfield; D. M. Nolan; H. G. Smith, Treasurer; Prof. D. W. Western, faculty representative.

The members of the Committee for the year 1948–49 are: Philip Curtis, Jr., chairman; Joseph Coudele, Eleanor Mansfield, Alexander Marshall, Ann Patterson, Adeline Petke, Rita Sisson, Chester Thomas, and Loren Wood.

Kappa Mu Epsilon, College of St. Francis

The *Illinois Delta* Chapter of *Kappa Mu Epsilon* records the following papers for 1947–48:

Fermagoric triangles, by Sister Mary Claudia O.S.F.
Attacks on mathematics and how to meet them, by Sister Mary Elizabeth O.S.F.
Macheroni's constructions, by Sister M. Rita Clare, O.S.F.
Calculating prodigies, by Mary Jean Lafond
Cryptographs and ciphers, by Patricia Young
Three fallacies, by Marie Crawley
History and trancendence of pi, by Betty Lanaue
Comparisons of astronomy and the calendar, by Anne Hutchings
The Pythagorean theorem and its extensions, by Lillian Rafac
Kirkman's "School-girl" problem, by Mary Lou Hodor
Gamma function, $0!=1$, by Jane Rourke
Pedal triangles, by Eleanor O'Connor.

The books, *Schoolmasters Assistant*, by Thomas Dilworth and *Mathematics and Religion*, by Sister Noel Marie were reviewed by Betty Frieburg and Sister Mary Hilary O.S.F.

The officers for 1947–48 were: President, Eleanor O'Connor; Vice-President, Elizabeth Lanoue; Secretary, Jane Rourke; Treasurer, Mary Jean Lafond; Faculty Advisor; Sister M. Rita Clare, O.S.F.

The officers for 1948–49 are: President, Jane Rourke; Vice-President, Mary Jean Lafond; Secretary, Lillian Rafac; Treasurer, Rita Groban.

Graduate Mathematics Club, Indiana University

During the second year of its existence, the *Graduate Mathematics Club* heard a series of talks by members of the mathematics department and two speakers from other departments. The topics were:

The arithmetic progression theorem, by Dr. G. W. Whaples
Bracketing and the ping-pong problem, by Dr. W. S. Gustin
Applications of linear algebra in quantum mechanics, by Dr. E. Konopinski
The nature of mathematical proof, by Dr. L. M. Graves
Bounded functions, by Dr. M. A. Zorn
Some statistical problems in bacteriophage work, by Dr. R. Dulbecco.

The Executive Committee for the year 1947–48 was: Mrs. Lee Pruett, I. N. Herstein, and Gordon Overholtzer.

The new Executive Committee is: Mrs. Lee Pruett, Joseph Sullivan, and David van Tijn.

Pi Mu Epsilon, Bucknell University

The program for the year 1947–48 of *Pennsylvania Beta* Chapter of *Pi Mu Epsilon* was:

The mathematics of jet propulsion, by H. I. Holman
The toss of a coin, by F. S. McFeely, Jr.
Ocean waves, by Prof. H. V. Flinsch

The history and computation of pi, by Mrs. W. E. Elze

Number, the language of science, by Marguerite Muller.

Officers for 1947–48 were: Director, Prof. D. P. Souders; Vice-Director, J. W. Sprout; Secretary, Mrs. W. E. Elze; Treasurer, Marguerite Muller.

Officers for 1948–49 are: Director, Prof. L. M. Swartz; Vice-Director, J. A. Bortner; Secretary, M. F. Nightingale; Treasurer F. S. McFeely, Jr.

Pi Mu Epsilon, Lehigh University

The *Pennsylvania Gamma* Chapter of *Pi Mu Epsilon* was reactivated in November 1947. Among the papers to be read at meetings were:

Certain aspects of statistics, by Prof. Garrison

Nomography, by Layton Butts

Topics in the theory of relativity, by Prof. K. W. Lamson

Some mathematical problems, by Prof. V. F. Cowling, a series of problems challenging the ability of the members.

Steps were taken to cooperate more closely with the *Newtonian Mathematics Society* for increasing the interest and scholarship in mathematics at Lehigh University.

Nine undergraduates and one faculty member were initiated in April at which occasion Prof. J. B. Reynolds, the retiring head of the department, was guest of honor. He gave an account of the progress and amusing events he has seen take place in the more than forty-one years he has been associated with Lehigh University.

The officers serving during 1947–48 were: Director, Prof. V. F. Cowling; President, Valerio Hunt; Secretary, J. H. Vogelsong; Treasurer, J. S. Adam.

The officers elected are: Director, Prof. R. R. Stoll; President, J. F. Ahern; Secretary, K. E. Ferree; Treasurer, L. E. Butts.

Mathematics Club, Ball State Teachers College

The following papers were presented to the *Mathematics Club* during 1947–48:

Topics from algebra, by James Swinford

Opportunities in statistics, by Merle Guthrie

Topics in trigonometry, by Kenneth Poucher

Paradox lost, by Kenneth Conkling

Applications of complex numbers in alternating currents, by Dr. Hummel

Vector analysis, by Elmo Purlee

Non-Euclidean geometry, by Dwain Small

Computational short-cuts, by Robert Dillon

Mathematics for the imagination, by Lewis Ward, Jr.

The annual picnic was held in May.

Officers who served during 1947–48 were: President, Elmo Purlee; Vice-President, Charles Galancy; Program Chairman, William Flora; Secretary, Edward Shreve; Treasurer, Nell Young.

Officers elected for 1948–49: President, John Pruden; Vice-President, Richard Moore; Secretary, Doris Mooney; Treasurer, Gale Brown.

Kappa Mu Epsilon, Hofstra College

The papers presented to the *New York Alpha* Chapter of *Kappa Mu Epsilon* for the year 1947–48 include:

The four-color problem, by Prof. L. F. Ollmann

Our nearest star, by Francis Wilson

A new series and some useless limits, by G. B. Charlesworth

Some properties of primes, by W. L. Marshall

A comparison of Oxford with American Universities, by Prof. Banesh Hoffmann, of Queens College

A short history of Kappa Mu Epsilon, by E. Marie Hove.

Following the second program the club visited the Hayden Planetarium. A Square Dance was held just before the Christmas vacation. The address by Dr. Hoffmann was given at the annual Initiation Banquet when 21 new members were initiated.

A prize was given to Fred Bloschies as the best student of Freshman Mathematics.

Officers for the year 1947–48 were: President, W. L. Marshall; Vice-President, Samuel Williams; Secretary, Gertrude Decker; Treasurer, J. G. Lutz; Corresponding Secretary, Frank Hawthorne; Faculty Sponsor, E. Marie Hove.

The officers elected for 1948–49 are: President, David Jordan; Vice-President, Richard Krause; Secretary, Gladys Meyer; Treasurer, Salvatore Cannizzaro; Corresponding Secretary, Beatrice Rohr; Faculty Sponsor, E. Marie Hove.

Pi Mu Epsilon, Michigan State College

During the year student and faculty speakers were alternated. The following topics were presented:

Mathematical problems in geography, by Joyce Clark

An envelope of pedal lines of a triangle, by Dr. Stewart

A two-variable maximum problem, by Joyce Deisch and Eugene Parker

Using I. B. M. *cards to find prime numbers*, by Prof. Frame

A theorem on partitions, by Charles Kraft

A theorem in continuous functions, by Cliff Gray

Radar equations, by Dr. Bell

A statistics program, by Dale Hekhuis and Robert Zabel.

Eleven members were initiated at the spring term and twenty at the fall term initiation. Since some members graduated, the total active membership is now thirty-four students.

In addition to the regular meetings, picnics were held in the fall and spring terms. At the annual banquet the L. C. Plant awards for deserving mathematics students were won by James Powell (first prize of $50) and Richard Zindler (second prize of $40). Prof. C. C. MacDuffee of the University of Wisconsin spoke on the topic *Is mathematics important?*

Officers for the year 1947–48 were: President, Eugene Parker; Vice-Presi-

dent, Nancy Waldo; Secretary, Joyce Clark; Treasurer, James Powell; Faculty Advisor, Dr. C. P. Wells.

Mathematics Club, Carleton College

The *Mathematics Club* of Carleton College has completed its second year of monthly meetings. Social activities included a picnic and a Christmas party with a program of mathematical recreations. The following talks were given during the year 1948–48:

Symbolic logic, by Dr. K. May

The mathematics of atomic collisions, by Dr. F. Vergrugge

Combinatorial topology, by Dr. G. K. Kalisch

Counting by dozens, by Prof. C. S. Carlson

Pythagorean numbers, by Mr. W. Crum.

Officers for 1947–48 were: President, Joan Shapper; Vice-President, Bob Henderson; Secretary, Jean Marie Baldwin; Faculty Advisor, Professor Kenneth May.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

ARMY RESERVE COMMISSIONS FOR MATHEMATICIANS AND STATISTICS SPECIALISTS

Reserve commissions in the Army of the United States are now available to qualified mathematicians and statistics specialists who meet the requirements of age, education, experience and physical condition. The commissions range from those of Second Lieutenant up to and including full Colonel, depending upon the qualifications and experience the mathematician or statistical specialist possesses in his specified field. Applicants must be at least twenty-one (21) years of age for appointment in the Reserve Corps,and applications will be considered from individuals up to fifty-five (55) years. All must be citizens of the United States. No previous military experience is required.

A bachelor's degree in mathematics or statistics, or in a field based heavily on mathematics, such as physics, engineering, economics, banking and finance, and accounting, qualifies for appointment into the Officers' Reserve Corps. Accepted applicants will be appointed in the grade of Second Lieutenant for service as mathematicians and statistics specialists in the Army Security Reserve; Medical Service Corps Reserve; and the Military Intelligence Reserve.

For appointments in grades of First Lieutenant through and including Colonel, applicants, in addition to the educational degree specified above, must have had qualifying education or progressive, professional experience in any one or any combination of the areas shown below. The Reserve sections for which

appointments in the grade of First Lieutenant and above will be made are: Army Security Reserve; Chemical Corps Reserve; Finance Department Reserve; Medical Service Corps Reserve; Military Intelligence Reserve; Quartermaster Corps Reserve; Signal Corps Reserve; and Staff and Administrative Reserve.

Six specific areas of experience may be offered by an applicant as meeting the minimum experience qualifications. These areas are: (1) Analysis; (2) Algebra; (3) Geometry; (4) Probability and Statistics; (5) Applied mathematics; (6) Other specified areas such as theory of numbers, methods of computation, combinatorial analysis, biometry, econometrics, and market research.

The above experience may be in any one or any combination of the following activities: (1) Professional practice in a field of applied mathematics; (2) Teaching; (3) Research; (4) Consulting; (5) Technical writing and editing.

Determination of the grade in which an applicant is to be commissioned is the function of the examining board which interviews the applicant, subject to final approval of The Adjutant General. The grade will be based on the total number of years of qualifying college education and/or experience according to the following scale of minimum requirements for each grade. For commission as a Second Lieutenant, the combined qualifying college education and/or experience shall amount to a minimum of four years. Qualifying college education and/or experience for commissioning as First Lieutenant shall equal a minimum of seven years; for Captain, eleven years; for Major, sixteen years; for Lieutenant Colonel, twenty-one years; and for Colonel, twenty-six years.

Full details of the program for commissioning of civilian mathematicians and statistics specialists are provided in Department of the Army Circular No. 210, dated 14 July 1948. Application forms and Circular No. 210 may be obtained from local Reserve Unit Headquarters or from Organized Reserve Unit Instructors, or by writing to Army Headquarters at New York, Baltimore, Atlanta, Chicago, Houston or San Francisco. Inquiries and requests for application forms and the circular may also be addressed to The Adjutant General, Department of the Army, Washington 25, D. C.

PERSONAL ITEMS

Professor I. S. Sokolnikoff of the University of California at Los Angeles has been awarded the Presidential Certificate of Merit for "outstanding contributions in aid of the war effort against the common enemies of the United States and its allies in World War II." Also, Professor Sokolnikoff has received a grant from Research Corporation, New York City, in support of the research project entitled "Two-dimensional problems in the theory of elasticity."

At Marshall College a Mathematics Conference of the elementary, secondary, and college teachers of West Virginia will be held on March 18–19, 1949. The purpose of the conference is to arouse interest in the improvement of the training of teachers.

The University of Akron announces the promotion of Mr. Louis Ross to an

assistant professorship and the appointment of Miss Edith E. Robbins as a part-time instructor.

University of Alabama announces the following appointments to instructorships: Miss Margaret Berger of Wagner College, Mr. Louis Jaffe of Ohio State University, Mrs. A. M. Jones of Mississippi Southern College, Mr. Hassel Palmer, and Mrs. Nell M. Reid.

University of Arizona makes the following announcements: Assistant Professor D. L. Webb has been promoted to an associate professorship; Assistant Ruth A. Fish has been promoted to an instructorship; Miss Phyllis Gray, assistant at University of Wisconsin, Miss Rose A. Grundman, graduate student at Northwestern University and Mr. D. B. Marsh, graduate student of the University of Arizona, have been appointed to instructorships; Miss Shirley A. Samuelson has been appointed to an assistantship. Professor R. F. Graesser, head of the Department of Mathematics, has been elected Chairman of the Southwestern Section of the Association and Professor E. J. Purcell has been elected Vice-Chairman of the Section.

University of California at Berkeley announces the following: Assistant Professor R. H. Sciobereti has been promoted to an associate professorship; Dr. E. W. Barankin has been promoted to an assistant professorship; Dr. T. M. Apostol, Dr. J. L. Hodges, Jr., Mr. T. A. Jeeves, Professor Michel Loeve of the University of London, Dr. S. F. Neustadter, Dr. Stephan Peters, and Miss Elizabeth L. Scott have been appointed Lecturers.

The State College of Washington announces the following appointments to instructorships: Jessie V. Allhands, Tyler Allhands, W. R. Ballard, Adalene M. Bibby, Celia E. Klotz and E. L. Salisbury.

The University of Kentucky makes the following announcements: Associate Professor Tadeusz Leser, Emory and Henry College, has been appointed to an assistant professorship; Mr. R. E. Wheeler has been appointed to an instructorship; Mr. E. C. Steele has received an appointment as graduate assistant; Miss Virginia Baskett and Mr. Wimberly Royster have been promoted to instructorships.

The University of Maine reports: Professor N. R. Bryan has retired with the title of Professor Emeritus; Mr. P. C. Rogers and Miss Joanne M. Springer have been appointed to instructorships.

The University of Massachusetts announces: Professor F. C. Moore, head of the Department of Mathematics, has retired with the title of Professor Emeritus; Assistant Professor A. E. Andersen has been appointed Professor and Head of the Department; Mr. S. I. Allen, formerly principal of the Graham-Eckes School, Palm Beach, and Mr. P. D. Ritegr, who has been Mathematics Master, The Phelps School, Malvern, Pennsylvania have been appointed to instructorships.

At the University of Miami, Mr. F. E. Adams and Mr. Ira Rosenbaum have been appointed to assistant professorships: Assistant Professor Robert Tindall has been appointed Assistant Professor of Engineering.

The University of Michigan announces the following: Professor L. C. Karpinski has retired with the title of Professor Emeritus; Associate Professor J. R. Britton of the University of Colorado has been appointed Visiting Associate Professor; Mr. R. K. Ritt, Dr. W. R. Scott, Dr. R. G. Stanton and Dr. W. R. Utz have been appointed to instructorships.

The Department of Mathematics of the University of Minnesota reports: Dr. B. B. Gelbaum and Dr. E. B. Nering of Princeton University have been appointed to assistant professorships; Dr. R. E. Graves has been promoted to an assistant professorship.

The Department of Mathematics and Mechanics of the University of Minnesota announces: Professor C. A. Herrick has retired with the title of Professor Emeritus; Associate Professors H. A. Doeringsfeld and F. E. Miller have been promoted to professorships; Assistant Professors J. C. Sanderson and R. W. Siler have been promoted to associate professorships; Assistants P. G. Kirmser and O. M. Rye have been promoted to instructorships; Mr. R. R. Johnson, Jr., Mr. L. G. Kelly and Mr. K. M. McMillin have been appointed to instructorships; Mr. G. C. Francis, Mr. A. R. Fredriksen, Mr. J. Joseph and Mr. J. Rynning have been appointed to teaching assistantships.

The University of Mississippi makes the following announcements: Professor T. A. Bickerstaff has been promoted to the chairmanship of the department; Associate Professor G. R. Trott has been promoted to a professorship; Mr. A. H. Samuels has been promoted to an assistant professorship; Mr. N. A. Childress and Mrs. Corrie D. Quarles have been appointed to instructorships. The annual meeting of the Louisiana-Mississippi Section of the Association will be held at the University of Mississippi, as a part of the centennial program of the University on April 8–9, 1949. Professor Saunders MacLane will be guest speaker. Special recognition will be given Dr. Alfred Hume, a charter member of the Association.

At the University of Missouri, Assistant Professor F. O. Duncan has retired; Assistant Professor Choy-tak Taam of Canton, China, has been appointed to an assistant professorship.

The University of Nebraska announces the promotions of Dr. Edwin Halfar and Dr. W. G. Leavitt to assistant professorships and the appointment of Miss F. Marion Clarke, formerly lecturer at the University of Southern California, to an instructorship.

The University of New Hampshire reports: Assistant Professor W. L. Kichline has been promoted to an associate professorship; Miss Elizabeth J. Fraser and Mr. Bernhart Snyder have been appointed to instructorships; Assistant Professor A. R. Harvey is now Research Fellow at California Institute of Technology.

At the University of North Carolina, Professor Archibald Henderson, formerly head of the Department of Mathematics, has retired; Dr. W. M. Whyburn, formerly president of Texas Technological College, has been appointed

Head of the Mathematics Department; Mr. I. R. Hershner, Jr., of the University of Chicago has been appointed Lecturer.

The Woman's College of the University of North Carolina announces the retirement of Professor Cornelia Strong and the appointment of Miss Lila P. Walker to an assistant professorship.

The University of North Dakota reports: Mr. S. C. Simonson has been appointed to an instructorship; Associate Professor S. L. Mason is on sabbatical leave and is studying at the University of Washingon.

The University of Oklahoma announces: Dr. S. W. Reaves has retired with the title Professor Emeritus of Mathematics; Associate Professor Dora McFarland has been promoted to a professorship; Assistant Professors Arthur Bernhart and B. S. Whitney have been promoted to associate professorships; Associate Professor A. A. Grau of the University of Alabama has been appointed to an associate professorship; Assistant Professor W. A. Catenaro of the University of South Carolina and Mr. R. B. Deal, Jr., have been appointed to instructorships.

The University of Oregon announces the following appointments: Assistant Professor F. J. Massey of the University of Maryland to an assistant professorship; Miss Vivienne Odishaw of the University of Pittsburgh and Mrs. Constance Stevens, University of Oregon, to instructorships.

The University of Pennsylvania reports the following: Associate Professor I. J. Schoenberg has been promoted to a professorship; Professor A. S. Besicovitch of Trinity College, Cambridge, England has been appointed Visiting Professor; Dr. R. D. Anderson of the University of Texas and Mr. Russell Remage, University of Delaware, have been appointed to instructorships.

At the University of Rochester, Instructors Frederick Bagemihl and Horace Komm have been promoted to assistant professorships; Mr. Arthur Danese has been appointed to an instructorship; Mr. Eugene Trabka has received an appointment as part-time instructor.

The University of Tennessee announces the following appointments: Dr. K. E. Brown, formerly at Wagner College, to an associate professorship; Mr. E. C. Campbell as lecturer; Mrs. Zoe E. Albert to an instructorship; Mr. Kenneth Carman, Mr. Nickolas Heerema, Mr. Joe Parsons and Mr. Lan Hsing Yieh to graduate assistantships; Mr. Ralph Grimble to a teaching assistantship.

The University of Toledo reports: Professor O. L. Dustheimer of Baldwin-Wallace College has been appointed Professor of Astronomy and Mathematics; Dr. C. J. Blackall, recently at the University of Detroit, has been appointed Associate Professor of Mathematics; Professor H. L. Zeiders of Midland College has been appointed to an assistant professorship.

The University of Utah makes the following announcements: Professor E. W. Pehrson has been made Professor Emeritus; Mrs. Marian Dickman has been appointed Lecturer; Associate Professor C. J. Thorne is on leave for the year 1948–49 and is at the University of California at Los Angeles.

The University of Virginia announces the following appointments to instructorships: R. R. Bernard, H. F. De Francesco, F. T. Dresser, W. H. Johnson, R. H. Kasriel, D. B. Lowdenslager, W. E. Pace, Shirley A. Rubenstein, C. M. Spaulding, and D. C. Walker.

The following appointments and promotions have been made at the University of Wisconsin: Professor Max Dehn, Black Mountain College, has been appointed Visiting Professor; Professor L. C. Young, University of Cape Town, has been appointed to a professorship; Assistant Professor O. G. Owens of the University of Nevada has been appointed to an assistant professorship; Mr. C. C. Hsiung, Michigan State College, Mr. J. B. Kelly, Massachusetts Institute of Technology, and Mr. C. S. Yih, Iowa State College, have been appointed to instructorships; Associate Professor S. C. Kleene has been promoted to a professorship; Mr. R. E. Fullerton has been promoted to an assistant professorship.

Washington University announces the following appointments to graduate assistantships: O. A. Biberstein, P. J. Ceike, G. C. Cree, John Hoelzer, A. D. Martin, R. A. Moore, M. C. Palmer, D. R. Scholz, George Schriro, and J. Isabel Stewart.

At Western Michigan College, Associate Professor C. H. Butler has been promoted to a professorship; Assistant Professor William Halnon has been promoted to an associate professorship.

The following appointments to instructorships have been made at West Virginia University: Mr. Ralph Brown, Mr. Earl Crisler, Mrs. Helen Godfrey, Miss Helen Heater, Miss Helen Pollock and Miss Laurie Wear.

Yeshiva University announces the promotions of Dr. Henry Sisman to an assistant professorship and of Mr. Daniel Block to an instructorship.

Mr. M. W. Aylor of the University of Virginia has been promoted to an assistant professorship in the Department of Engineering Mathematics.

Dr. H. G. Booker, lecturer in mathematics at Cambridge University and supervisor of applied mathematics in Christ's College, has been appointed Professor of Electrical Engineering at Cornell University.

Reverend J. C. Burke, C.S.C., of the University of Notre Dame has been promoted to an assistant professorship.

Associate Professor J. A. Clarkson of the University of Pennsylvania has been appointed Head of the Department of Mathematics at Tufts College.

Mr. R. T. Donnell, University of Tennessee, has been appointed to an instructorship at Union College, Jackson, Tennessee.

Mrs. Rosalie L. Dunham, director of mathematics at Okennegee Junior College, has received an appointment as acting instructor at the University of Tulsa.

Miss Gertrude Foster of the University of South Carolina has retired.

Professor R. E. Gaines of the University of Richmond has retired with the title of Professor Emeritus.

Assistant Professor N. C. Hunsaker of Utah State Agricultural College has been promoted to an associate professorship.

Miss Dorothy Ingram, who has been teaching at New Armijo School, New

Mexico, has been appointed to an instructorship at the University of New Mexico.

Miss Rosamond Jones, University of Wisconsin, is teaching at Ely Junior College, Ely, Minnesota.

Mr. George Lawler of the University of North Dakota has received an appointment at Coalinga Junior College, Coalinga, California.

Dr. A. O. Lindstrum, Northwestern University, has been appointed to an assistant professorship at Knox College.

Mr. Hugo Mandelbaum has been appointed to an instructorship at Wayne University.

Mr. S. S. McNeary of Drexel Institute of Technology has been promoted to an associate professorship.

Dr. L. L. Merrill, formerly consulting mathematician in the research department of Stromberg-Carlson, Rochester, has been appointed Director of Graduate Studies and Professor of Mathematics at Clarkson College of Technology.

Dr. W. H. L. Meyer, Jr. of the College of the University of Chicago has been promoted to an assistant professorship.

Mr. C. R. Newell has been appointed to an instructorship at Niagara University.

Mr. J. J. Rowland, State College of Washington, has accepted a research position in the Engineering Department of the Boeing Airplane Company, Seattle.

Mr. W. J. Scharf has been appointed to an instructorship at John Carroll University.

Dr. J. A. Schouten has been appointed to a professorship at the University of Amsterdam.

At the University of Wyoming, Associate Professor Nathan Schwid has been promoted to permanent tenure.

Associate Professor D. R. Shreve of the University of Tulsa has accepted a position as research mathematician with the Carter Oil Company, Tulsa.

Dr. Mary-Elizabeth L. Solari has been appointed Lecturer at Chelsea Polytechnic, London, England.

Dr. G. L. Tiller of the University of Kentucky has been appointed to an assistant professorship at Utica College of Syracuse University.

Mr. A. C. Baird of Hamilton, Ohio died October 13, 1948. He was a charter member of the Association.

Dr. J. T. Rorer of Temple University died August 13, 1948.

Professor Emeritus C. T. Sullivan of McGill University died September 17, 1948.

Dean R. C. Tolman of the Graduate School, California Institute of Technology, died September 5, 1948.

Associate Professor Emeritus J. W. A. Young of the University of Chicago died October 26, 1948. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Report and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred thirty-six persons have been elected to membership by the Board of Governors on applications duly certified:

DON ALKIRE, M.A. (South Dakota) Asst. Professor, Mankato State Teachers College, Mankato, Minn.

J. E. ALMAN, M.A. (Claremont) Instructor, College of Liberal Arts, Boston University, Mass.

J. S. ARONOFSKY, M.S. (Stevens Inst.) Research Engineer, Westinghouse Research Labs., Pittsburgh, Pa.

P. R. BEESACK, Student, McMaster University, Hamilton, Ontario, Canada

ALICE K. BELL, M.A. (Michigan) Asso. Professor, Fresno State College, Calif.

LEON BENSON, A.B. (New York) Instructor, Lehigh University, Bethlehem, Pa.

A. F. BENTLEY, Ph.D. (Johns Hopkins) R. 2, Paoli, Ind.

JAMES BERCOS, B.S. (Illinois) Instructor, University of Georgia, Athens, Ga.

R. R. BERNARD, M.A. (Virginia) Instructor, University of Virginia, Charlottesville, Va.

D. W. BLACKETT, M. A. (Princeton) Graduate Student, Princeton University, N. J.

A. P. BOBLETT, B.S. (Western Reserve) Instructor, Kent State University, Ohio

A. B. BOGGS, M.A. (Michigan) Asst. Professor, Michigan College of Mining & Technology, Houghton, Mich.

W. L. BOYD, B.S. (Oklahoma) Graduate Asst., Oklahoma University, Norman, Okla.

HELEN J. BRADLEY, B.A. (Rosemont) Instructor, University of Tennessee, Knoxville, Tenn.

MILDRED J. BRANNON, M.A. (Illinois) Asst. Professor, Michigan State Normal College, Ypsilanti, Mich.

W. C. BROWN, B.S. (Colorado A & M) Graduate Asst., University of Oklahoma, Norman, Okla.

F. E. BRUNTZ, M.A. (Colorado State College) Asst. Professor, University of Denver, Colo.

REV. V. J. BURKE, T.O.R., M.A. (St. Francis) The College of Steubenville, Ohio

MRS. E. R. BUTLER, B.S. (Iowa State) Instructor, Iowa State College, Ames, Iowa

W. H. CLATWORTHY, M.A. (Kentucky) Instructor, Wayne University, Detroit, Mich.

L. A. COLQUITT, M.A. (Ohio State) Asst. Professor, Texas Christian University, Fort Worth, Tex.

ESTHER A. COMPTON, M.A. (Indiana) Instructor, Cumberland College, Williamsburg, Ky.

W. F. CORNELL, M.A. (Ohio State) Asso. Professor, Bowling Green State University, Ohio

JANE S. CRONIN, Ph.D. (Michigan) ONR Fellow, Princeton University, N. J.

A. E. DANESE, A.M. (Harvard) Instructor, University of Rochester, N. Y.

H. A. DOERINGSFELD, C.E. (Wisconsin) Professor, University of Minnesota, Minneapolis, Minn.

F. T. DRESSER, B.S. (Virginia Military Inst.) Part-time Instructor, University of Virginia, Charlottesville, Va.

MRS. ESTHER S. DUNKELBERGER, M.Litt. (Pittsburgh) Instructor, Duquesne University, Pittsburgh, Pa.

O. L. DUSTHEIMER, Ph.D. (Michigan) 4935 East 71st St., Cleveland, Ohio

REV. A. J. EIARDI, S.J., M.S. (Boston College) Asso. Professor, Boston College, Chestnut Hill, Mass.

W. R. EIKELBERGER, M.S. (Fort Hays) Asst. Professor, University of Denver, Colo.

J. L. EMMERT, M.Ed. (Pittsburgh) Asst. Professor, The College of Steubenville, Ohio

J. L. ERICKSEN, B.S. (Washington) Teaching Fellow, Oregon State College, Corvallis, Ore.

G. W. FAIRCHILD, Student, Arizona State College, Tempe, Ariz.

RUTH A. FISH, M.S. (Arizona) Instructor, University of Arizona, Tucson, Ariz.

PEARL L. FORD, A.M. (Michigan) Asst. Professor, Western Michigan College of Education, Kalamazoo, Mich.

W. B. FULKS, M.A. (Arkansas) Instructor, University of Minnesota, Minneapolis, Minn.

J. R. GARRETT, A.M. (Duke) Visiting Instructor, Duke University, Durham, N. C.

R. E. GETTIG, M.S. (Pittsburgh) Instructor, University of Pittsburgh, Pa.

E. W. GOINGS, M.S. (Michigan) Asst. Professor, Michigan State Normal College, Ypsilanti, Mich.

D. A. GORSLINE, B.S. (Hendrix) Graduate Asst., University of Oklahoma, Norman, Okla.

H. E. GOULD, M.A. (New York) Instructor, Rhode Island State College, Kingston, R. I.

D. O. GRAY, M.B.A. (Houston) Instructor, University of Houston, Tex.

L. D. GREGORY, B.A. (Oklahoma) Instructor, University of Oklahoma, Norman, Okla.

MRS. BLANCHE B. GROVER, M.A. (Texas) Asst. Professor, University of Houston, Tex.

ROSE A. GRUNDMAN, M.S. (Northwestern) Instructor, University of Airzona, Tucson, Ariz.

FRANKLIN HAIMO, Ph.D. (Harvard) Asst. Professor, Washington University, St. Louis, Mo.

E. E. HASKINS, Ph.D. (Boston) Asso. Professor, Fenn College, Cleveland, Ohio

MARQUERITE Z. HEDBERG, Ph.D. (Missouri) Adjunct Professor, University of South Carolina, Columbia, S.C.

M. S. HENDRICKSON, Ph.D. (Ohio State) Asso. Professor, University of New Mexico, Albuquerque, N. M.

RUTH E. HENNING, M.A. (Illinois) Instructor, Virginia Junior College, Minn.

REV. J. C. HILFERTY, B.A. (St. Charles) Instructor, St. Thomas More High School, Philadelphia, Pa.

J. R. HODGES, B.S. (Tulane) Teaching Fellow, Tulane University, New Orleans, La.

M. R. HORAK, M.A. (Michigan) Vice Principal, Randolph Central School, N. Y.

C. C. HSIUNG, Ph.D. (Michigan State) Instructor, University of Wisconsin, Madison, Wis.

K. C. HSU, B.S. (Chiao-Tung University) Graduate Student, University of Kansas, Lawrence, Kan.

S. L. HULL, M.S. (Iowa) Instructor, University of Arkansas, Fayetteville, Ark.

W. H. JOHNSON, B.A. (Virginia) Instructor, University of Virginia, Charlottesville, Va.

F. E. JUSTIS, M.S. (West Virginia) Asst. Professor, Geneva College, Beaver Falls, Pa.

W. C. KALINOWSKI, Ph.D. (Saint Louis) Asst. Professor, St. John's University, Collegeville, Minn.

GILBERT KASKEY, M.A. (Delaware) Instructor, University of Delaware, Newark, Del.

R. H. KASRIEL, B.S. (Tampa) Graduate Student, University of Tampa, Fla.

B. C. KENNY, B.S. (Bethany) Graduate Asst., Lehigh University, Bethlehem, Pa.

MAX KRAMER, M.A. (Columbia) Instructor, University of Illinois, Champaign, Ill.

CHARLES KURLAND, B.A. (Buffalo) Student, University of Buffalo, N. Y.

ROBERT LANKTON, M.A. (Wayne) Asst. Professor, Iowa State Teachers College, Cedar Falls, Iowa

M. L. LAWSON, M.S. (Oklahoma A & M) Instructor, University of Oklahoma, Norman, Okla.

MARY B. LIEBERKNECHT, B.S. (Iowa State) Instructor, Iowa State College, Ames, Iowa

R. K. LONGLEY, B.A. (Buffalo) Head of Dept., Canisteo Central School, N. Y.

HUGO MANDELBAUM, Dr. rer. nat. (Hamburg) Instructor, Wayne University, Detroit, Mich.

MARY O. MARQUARDT, M.A. (Illinois) Asst. Professor, Industrial Labor Relations School, Cornell University, Ithaca, N. Y.

D. C. B. MARSH, JR., M.S. (Arizona) Instructor, University of Arizona, Tucson, Ariz.

J. E. MARTIN, M.S. (Vanderbilt) Asst. Professor, Davidson College, N. C.

M. G. MCCUAN, M.A. (West Texas State) Asso. Professor, Amarillo College, Tex.

PAUL MEIER, M.A. (Princeton) Asst. Professor, Lehigh University, Bethlehem, Pa.

D. M. MESNER, B.A. (Nebraska) Graduate Student, Northwestern University, Evanston, Ill.

J. W. METTLER, M.A. (Bucknell) Instructor, Lehigh University, Bethlehem, Pa.

P. T. MIELKE, M.Sc. (Brown) Industrial Research Fellow, Purdue University, West Lafayette, Ind.

ROSEMARY MILKOVITCH, M.A. (Montana) Instructor, Bemidji State Teachers College, Minn.

P. D. MINTON, M.Sc. (Southern Methodist) Instructor, University of North Carolina, Chapel Hill, N. C.

LEONID MIRSKY, M.Sc. (London) Lecturer, University of Sheffield, England

R. D. MORRISON, M.S. (Oklahoma A & M) Asst. Professor, Oklahoma A & M College, Stillwater, Okla.

RUTH E. MUSE, M.A. (Minnesota) Instructor, Lincoln University, Jefferson City, Mo.

SAM NAIDITCH, Ph.D. (Calif. Inst. of Tech.) Asst. Professor, Duquesne University, Pittsburgh, Pa.

A. C. NELSON, JR., M.S. (Delaware) Instructor, University of Delaware, Newark, Del.

MARY NELSON, M.S. (State Univ. of Iowa) Asst. Professor, Utah State Agricultural College, Logan, Utah.

PAUL NESBEDA, Ph.D. (Pisa) Instructor, Catholic University, Washington, D.C.

COL. C. P. NICHOLAS, B.S. (U.S.M.A.) Professor, U. S. Military Academy, West Point, N. Y.

DOROTHY M. OEHMKE, B.S. (Detroit) Graduate Student, University of Detroit, Mich.

R. H. OEHMKE, B.S. (Michigan) Instructor, University of Detroit, Mich.

R. R. OTTER, Ph.D. (Indiana) Asst. Professor, University of Notre Dame, Ind.

O. E. OVERN, Ph.D. (Columbia) State Teachers College, Milwaukee, Wis.

HASELL PALMER, M.S. (Tennessee) Instructor University of Alabama, University, Ala.

D. K. PARKS, A.B. (Denver) Instructor, University of Denver, Colo.

M. O. PEACH, M.S. (Carnegie) Instructor, Carnegie Institute of Technology, Pittsburgh, Pa.

H. C. PETERSON, M.A. (Denver) Instructor, University of Denver, Colo.

P. B. PEYTON JR., B. Engg. (Virginia) Asst. Professor, Davidson College, N. C.

MRS. ROBERTA E. PRESNELL, M.A. (Beloit) Instructor, Rockford College, Ill.

GORDON RAISBECK, B.A. (Stanford) Instructor, Massachusetts Institute of Technology, Cambridge, Mass.

S. E. RAUCH, Ph.D. (Stanford) Asso. Professor, University of California, Santa Barbara College, Calif.

J. M. REUBER, M.S. (Chicago) Asst. Professor, College of St. Thomas, St. Paul, Minn.

A. A. RITCHIE, M.S. (Oklahoma A & M) Instructor, Oklahoma A & M College, Stillwater, Okla.

GORDON RITCHIE, Student, McMaster University, Hamilton, Ontario, Canada

EDITH E. ROBBINS, B.S. (Akron) Instructor, University of Akron, Ohio

LT. COMDR. V. N. ROBINSON, Ph.D. (Chicago) Asso. Professor. U. S. Naval Academy, Annapolis, Md.

C. A. ROGERS, M.S. (North Texas State Teachers) Asst. Professor, University of Houston, Texas

IRA ROSENBAUM, M.S. (Harvard) Asst. Professor, University of Miami, Fla.

T. S. ROSS, B.A. (Central State Teachers College) Part-time Instructor, University of Oklahoma, Norman, Okla.

R. P. ROWLEY, M.S. (Syracuse) Instructor, Millard Fillmore College, University of Buffalo, N. Y.

R. E. RUSH, B.S. (Arizona State College) Meteorologist, U. S. Weather Bureau, Tempe, Arizona

ARTHUR SAASTAD, M.S. (Northwestern) Asst. Professor, De Paul University, Chicago, Ill.

F. W. SAUNDERS, M.A. (North Carolina) Graduate Student, University of North Carolina, Chapel Hill, N. C.

C. H. SAVIT, M.S. (Calif. Inst. of Tech.) Mathematician, Western Geophysical Co. Los Angeles, Calif.

GEORGE SCHRIRO, B.S. (New York) Graduate Asst., Washington University, St. Louis, Mo.

H. A. SEEBALD, B.A. (Lehigh) Instructor, Lehigh University, Bethlehem, Pa.

SISTER MADELEINE R. ASHTON, M.S. (St. Louis) Instructor, College of the Holy Names, Oakland, Calif.

SISTER M. MICHAEL, M.S. (Catholic University), Instructor, Mt. Mercy College, Pittsburgh, Penna.

B. D. SMITH, B.Ch.E. (Minnesota) Instructor, University of Minnesota, Minneapolis, Minn.

T. C. SMITH, M.A. (New York) Asst. Professor, Youngstown College, Ohio

W. N. SMITH, M.A. (Northwestern) Asst. Professor, University of Wyoming, Laramie, Wyo.

J. M. STALEY, M.S. (Michigan) Instructor, Colorado A & M College, Fort Collins, Colo.

J. K. STEWART, Ph.D. (West Virginia) Asso. Professor, West Virginia University, Morgantown, W. Va.

R. R. STOLL, Ph.D. (Yale) Asso. Professor, Lehigh University, Bethlehem, Pa.

MRS. LOIS M. SUPROCK, B.A. (Toledo) Instructor, University of Toledo, Ohio

R. C. TALIAFERRO, Ph.D. (Virginia) Teacher, Portsmouth Priory School, Portsmouth, R. I.

MICHAEL TIKSON, B.S. (Youngstown) Graduate Asst., Lehigh University, Bethlehem, Penna.

J. L. ULLMAN, A.B. (Buffalo) Graduate Student, Stanford University, Calif.

J. J. VANDE CASTLE, B.S. (St. Norbert) Instructor, St. Norbert College, West De Pere, Wis.

J. H. WAHAB, B.S. (William & Mary) Fellow, University of N. Carolina, Chapel Hill, N. C.

D. C. WALKER JR., B.S. (Richmond) Instructor, University of Virginia, Charlottesville, Va.

LILA P. WALKER, M.A. (North Carolina) Asst. Professor, Woman's College of University of North Carolina, Greensboro, N. C.

WILLIAM WHITWAM, Chemical buyer, Park Grant Co., Watertown, S. D.

ARTHUR WORMSER, Dr. Engg. (Berlin) Engineer, Niehle Printing Press and Mfg. Co. Chicago, Ill.

JEAN M. WYRE, M.A. (Oberlin) Asst. Instructor, Berea College, Ky.

R. P. YIELDING, B.S. (U.S.M.A.) Instructor, University of Oklahoma, Norman, Okla.

J. A. ZILBER, A.M. (Harvard) Lecturer, Columbia University, N. Y.

SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Section was held at the United States Naval Academy, Annapolis, Maryland, on May 8, 1948, with Professor J. B. Scarborough presiding for the Chairman, Professor E. J. Finan, who was unable to attend.

There were one-hundred and fourteen persons attending the meeting including the following sixty-three members of the Association: H. C. Ayres, N. H. Ball, Archie Blake, J. A. Buikstra, R. S. Burington, H. C. Campaigne, S. R. Clements, Abraham Cohen, W. H. Cowles, G. F. Cramer, A. E. Currier, C. H. Denbow, J. A. Duerksen, Anselm Fisher, M. K. Fort, Jr., B. C. Getchell, Michael Goldberg, R. A. Good, J. R. Gorman, E. S. Grable, E. C. Gras, D. W. Hall, J. R. Hammond, B. I. Hart, G. C. Helme, M. A. Hyman, S. B. Jackson, W. Jennings, L. M. Kells, H. L. Kinsolving, W. D. Lambert, O. E. Lancaster, A. E. Landry, D. C. Lewis, G. A. Lyle, M. H. Martin, E. S. Mayer, C. V. McCamman, K. F. McLaughlin, Emanuel Mehr, Joseph Milkman, T. W. Moore, W. K. Morrill, W. H. Norris, J. F. Paydon, J. W. Popow, O. J. Ramler, C. H. Rawlins, Jr., R. W. Rector, J. N. Rice, R. E. Root, J. B. Scarborough, E. D. Schell, V. G. Schult, V. E. Spencer, H. C. Stilwell, O. M. Thomas, P. D. Thomas, J. A. Tierney, M. M. Torrey, W. R. Utz, C. H. Wheeler, H. S. Wilson.

The following papers were presented at the morning session:

1. *Some new algorithms*, by Captain N. A. Draim, United States Navy, introduced by the Secretary.

Algorithms for the extraction of square roots, the determination of the factors of composite numbers, and the solution of equations of the form $n^2+2an+b=\phi^2$ were presented.

2. *Recent advances in the Brun approach to the Goldbach problem*, by Dr. R. C. Rand, United States Naval Academy.

A summary of Buchstab's recent contributions to the Goldbach problem was presented. It was based on lectures of R. D. James at the Canadian Mathematical Congress in 1948 at Toronto. If $F(x, x^{1/\alpha})$ denotes the number of partitions of an even integer x into two positive integers both relatively prime to all primes less than $x^{1/\alpha}$, the Goldbach hypothesis will be established once it is shown that $F>0$ for $\alpha=2$. With the Bruin sieve it can be shown that

$$F(x, x^{1/\alpha}) = \lambda(\alpha)\frac{x}{\log^2 x} + O\left(\frac{x}{\log^3 x}\right)$$

where $\lambda>0$ for $\alpha\geqq 8$. Buchstab has shown that $\lambda(5)>0$, from which it follows that every sufficiently large even integer is the sum of two integers each containing at most four factors.

3. *On the number of solutions of the equation* $k\phi(n)=n-1$, by Dr. G. F. Kramer, and Dr. H. H. Campaigne, of the Navy Department.

According to D. H. Lehmer, if the equation $k\phi(n)=n-1$ (where $k>1$ is an integer and $\phi(n)$ is Euler's totient function) has a solution n which is composite, n must be of the form $n=p_1p_2\cdots p_r$, where the p_i are distinct odd primes and $r\geqq 7$. The question of the existence or non-existence of composite solutions of this equation was left unanswered, but it was shown that for a given integer $r>1$, there cannot be more than a finite number of composite solutions n having exactly r distinct prime factors when k is a positive integer.

4. *Roots of numerical equations by number sequences*, by Professor John Tyler and J. P. Hoyt of the United States Naval Academy, introduced by the Secretary.

The sequence $f(0), f(1), f(2), \cdots$ was used to find the rational roots and roots of the form $(a\pm\sqrt{b})/c$ of the equation

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

in which the coefficients a_i are integers. To each rational root there corresponds a subsequence constructed from the factors of $f(0), f(1), f(2), \cdots$ whose first order differences are constant. Corresponding to a root of the form $(a\pm\sqrt{b})/c$ there is a subsequence whose second differences are constant.

5. *The veteran and the accelerated course in mathematics*, by W. H. Norris of the Veteran's Center, Norfolk, Virginia.

A survey of the problems met in accelerated courses for veterans was followed by an appraisal of accelerated courses in the light of modern pedagogical thought.

6. *American mathematics turns useful*, by M. H. Slud of Catholic University.

The speaker dealt with the difficulties encountered by the engineer in attempting to reduce the problems in metallurgy to mathematical form. The various modern theories of brittleness, hardness, elasticity, and plasticity were summarized and compared.

M. H. MARTIN, *Secretary*

MAY MEETING OF THE SOUTHWESTERN SECTION

The Southwestern Section of the Mathematical Association of America met concurrently with the Southwestern Division of the A. A. A. S. on the campus of New Mexico Highlands University, Las Vegas, New Mexico, on Monday, May 3, 1948. Professor R. F. Graesser of the University of Arizona, Chairman of the Section, presided. Twenty-three persons were in attendance.

Officers elected for 1948–1949 were: Chairman, Earl Walden, of New Mexico College of Agriculture and Mechanical Arts; Vice-Chairman, E. J. Purcell, University of Arizona. Professor A. W. Boldyreff, University of New Mexico, was chosen as visiting lecturer for the year.

The program consisted of the following papers:

1. *Abstract isomorphism*, by Duane Studley, Colorado Springs, Colorado.

The relation of isomorphism was investigated at some length, and after consideration of the usual uses of the term certain generalizations were introduced. An abstract isomorphism was defined so as to contain the usual sense on specification. Some of the aspects of this consideration are new, and therefore some positive contribution was made.

2. *On Fermat's congruence* $A^{n-1}=1 \pmod{n}$ *for composite values of* n, by A. W. Boldyreff, University of New Mexico, Albuquerque, New Mexico.

3. *Extended analytic geometry applied to simultaneous equations*, by R. S. Underwood, Texas Technological College, Lubbock, Texas.

Given two simultaneous equations in three variables, a single equation in the three variables is found which is satisfied by all common solutions of the original pair, and whose locus is a curve coinciding with or containing the intersecting curve on a 3-axes plane. From this the common integral solutions may be found by a method similar to the one for simultaneous linear equations which was presented to this Section a year ago.

4. *Modern mathematics for everybody*, by Duane Studley, Colorado Springs, Colorado.

Part Six of the little book *Modern Mathematics for Everybody* entitled "Other Types of Set" was read. After opening remarks on the set concept, there were sections dealing with the topics domain, field, ring, algebra, and group. The simple exposition contained definitions and examples (those for groups being given at greatest length), and included the concepts of factor group and lattice. Cosets were clearly defined and the material was developed to the Galois level.

5. *Inverse variation problem*, by Lincoln La Paz, University of New Mexico.

This paper consisted of a systematic formulation and analysis of inverse variation problems, other than those connected with the Hamilton-Caratheodory function, by means of a generalization of the curvilinear congruence theory of differential geometry.

B. D. ROBERTS, *Secretary*

MAY MEETING OF THE NEBRASKA SECTION

The twenty-fourth meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, on Saturday, May 1, 1948. Professor H. M. Cox, Chairman of the Section, presided.

The following eleven members of the Association were present: W. C.

Brenke, C. C. Camp, Helen E. Clarkson, H. M. Cox, J. M. Earl, C. B. Gass, E. H. Hadlock, C. A. Huck, C. R. Perisho, H. L. Rice, and Lulu L. Runge.

At the business meeting the following officers were elected: Chairman, H. L. Rice, University of Omaha; Vice-Chairman, H. M. Cox, University of Nebraska; Secretary, Lulu L. Runge, University of Nebraska; Alternate, W. G. Leavitt, University of Nebraska; Governor, C. C. Camp.

The following papers were presented:

1. *On the decomposition of a space*, by Dr. Edwin Halfar, University of Nebraska, introduced by H. M. Cox.

In a topological space L defined by a derived set operator d satisfying $d(A+B)=dA+dB$, $D^2A=d(dA)\subset dA, dO=O$, such that for each $A\subset L$, there is a $DA\subset L$, one defines a maximal self-dense subset of A as the largest subset B of A satisfying $B\subset dB$. It is shown that the frontier of a set A has in general a maximal self-dense subset, and that this subset is decomposable into a sum of orthogonal self-dense subsets arising from the derived sets formed by operating with d on the isolated parts of the successive derived sets of the frontier of A.

2. *On the matrices whose elements are holomorphic functions*, by W. G. Leavitt, University of Nebraska.

This speaker considered similarity transformations (that is, transformations of type $T^{-1}AT$) of a matrix $A(z)$ each of whose elements is a function of the complex variable z. The results apply to a region R of the z-plane such that, in and on the boundary of R, each element of $A(z)$ and each of the roots of its characteristic equation is a holomorphic function. It was shown that there exists a matrix $T(z)$ whose determinant is non-vanishing in R and such that $T^{-1}AT$ is in a normal form with zeros below the main diagonal. It was first shown that the set of all functions holomorphic over R satisfies all the hypotheses of a principal ideal ring. It was therefore possible to use in the course of the proof certain abstract algebraic theorems known for such rings.

3. *On the deductive method*, by L. A. Rife, Philosophy Department, University of Nebraska, introduced by Professor Cox.

It was noted that the deductive method is a powerful tool in the fields of natural science as well as in philosophy. It is sometimes referred to as the methodology of deductive sciences or of mathematics. The value of a clear and concise understanding of the method was emphasized. Reference was made to *The Basis and Structure of Knowledge*, by W. H. Werkmeister. A lucid illustration, using only two axioms and other necessary principles for a model and interpretation of a deductive theory, was taken from Chapter VI of *Introduction to Logic*, by Alfred Tarski.

4. *Mathematics of finance*, by Professor C. A. Huck, Nebraska State Teachers College, Peru.

Professor Huck demonstrated the importance of the subject of mathematics of finance by illustrating some of its applications.

5. *The accuracy of computed results*, by E. Z. Palmer, Bureau of Business Research, University of Nebraska, introduced by Professor Cox.

This paper dealt with the usual case in which the numbers used in computation are inaccurate, the exact values being equally probable over the range between definite limits. The probability functions for the results of the addition, subtraction, multiplication, and division of two numbers were given, also those for the powers and roots of a number. The formulas for the mean and the standard deviation, and in some cases a general formula for the moments of these distributions, were given. It was pointed out that in the results of division, computation of powers, and root

extraction, the mean of all possible results was not identical with the central or usual integral result.

A suggestion for the proper statement of computed results was made, namely that the unit of accuracy of such a result should be taken as the integral power of ten which is immediately less than three times the standard deviation of the probability function of that result.

6. *On p-adic numbers*, by W. T. Lenser, University of Nebraska, introduced by Professor Cox.

This was an expository talk on the field of p-adic numbers as originally developed by Hensel, with particular reference to order relations and the addition and multiplication of certain complex numbers in p-adic form.

7. *On an experiment in teaching essentials of mathematics to deficient students*, by Dr. Edwin Halfar, University of Nebraska, introduced by Professor Cox.

8. *Practical applications of mathematics of certain aspects of naval activities*, Lieut. Commander G. E. Peddicord, USN, University of Nebraska.

Commander Peddicord discussed the problem of hitting an air target, the most difficult of all fire-control problems. The computer used to calculate the corrections for this type of problems does all of its calculating by mechanical basic mechanisms. It transmits and receives most of its signals electrically.

An air target position is established by three coördinates, namely, range, bearing and elevation. The computer must generate continuous values of range as a part of its many calculations. Ship and target motion are resolved into vectors in and across the line of sight by component solvers. The range component, in the line of sight, is due to ship motion and target horizontal and vertical motion. The range components are added in differentials and the sum multiplied by increments of time in a range or disk-type integrator. The product is then added to initial range in another differential, and thus continuous present range to the target is computed in the basic mechanisms of the computer.

9. *Exhibit of manuscript arithmetic handwritten in 1848 by James Riley Cox*, by H. M. Cox, Bureau of Instructional Research, University of Nebraska.

10. *Demonstration of punched card tabulating machines*, by H. M. Cox, University of Nebraska.

LULU L. RUNGE, *Secretary*

MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Union College, Schenectady, New York, on Saturday, May 1, 1948. The Chairman of the Section, Professor D. S. Morse of Union College, presided at the morning session; the Vice-Chairman, Professor E. B. Allen of Rensselaer Polytechnic Institute, presided at the afternoon session. At the conclusion of the afternoon session tea was served in the Faculty Lounge of Hale House.

About one hundred and fifteen persons were present, including the following sixty-three members of the Association: E. B. Allen, H. T. W. Aude, G. B. Banks, R. A. Beaver, W. W. Bessell, J. S. Biggerstaff, Harry Birchenough, W. J. Bruns, F. J. H. Burkett, K. A. Bush, S. S. Cairns, I. S. Carroll, W. B. Carver, F. F. Decker, E. J. Downie, W. C. G. Fraser, H. M. Gehman, B. H. Gere, Lillian Gough, J. B. Freeley, W. L. Greene, Lucille Hetzelt, H. K. Holt, M. E. Jenkins,

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The following officers were elected: Chairman, E. B. Allen, Rensselaer Polytechnic Institute; Vice-Chairman, W. H. Durfee, Hobart College; Secretary, C. W. Munshower, Colgate University. The 1949 meeting will be held at the University of Buffalo in the spring of 1949.

The following papers were presented:

1. *Auxiliary tricks which make mathematical notions stick*, by Reverend Benjamin Kuhn, Siena College.

The speaker made suggestions with respect to aiding the memories of mathematics students.

2. *Complex geometry*, by Professor B. C. Patterson, Hamilton College

Observations and comments were made on the analytic study of geometries in the complex plane, with particular reference to the representation of points, the representation of curves and other varieties, and the algebraic and differential invariant theories.

3. *Non-planar graphs*, by Dr. Bryant Tuckerman, Cornell University.

Following introductory remarks about non-planar graphs, several possible applications with respect to electrical circuits were indicated.

4. *The history of topics in analytic geometry*, by Professor Vera Sanford, Oneonta State Teachers College.

The speaker commented upon some of the outstanding contributions to analytic geometry commencing with Descartes and Fermat.

5. *Some models useful in the teaching of high school and college mathematics*, by Professor Ruth Stokes, Syracuse University.

Professor Stokes exhibited at least one model illustrating some figure from almost every branch of mathematics, and offered suggestions for the use of such models in teaching. In particular, she exhibited models constructed of plastic materials to illustrate propositions in solid geometry, analytics, and the calculus; a double cone string model for use with a projecting lantern and slides for showing the conic sections; Klein's bottle made of glass, illustrating a one-sided surface; and, for illustrating the fourth and higher dimensions, she showed stick models, of balsa wood, of hyperpyramids and hypercubes.

6. *Oriented elements in analytic geometry*, by Professor W. J. Bruns, Syracuse University.

It was contended that even in teaching the very elements of analytic geometry, three details should not be neglected: (1) any true geometric property should be presented as an invariant under displacements, or, what is the same, under coordinate transformations; (2) only purely analytic processes should be permissible; (3) any proof of a formula or a theorem should be independent of the accidental position of the parts of a figure and of the position of the figure relative

to the coordinate system. These demands are frequently violated in our textbooks. One way to avoid such shortcomings is to use oriented elements such as the oriented line or "spear," the oriented plane, the oriented angle between non-oriented lines as well as between oriented ones, and the oriented polygon. Several illustrations were given by the speaker, who demonstrated in detail how difficulties encountered in the usual treatment of the normal form of the straight line vanished when oriented elements were introduced.

7. *Monotonic transformations*, by Professor Edith Schneckenburger, University of Buffalo.

Topological properties of monotonic transformations were discussed. Emphasis was put upon invariant properties and conditions resulting in topological equivalence. Some suggestions were made regarding the use of this transformation in other geometries.

8. *Applied mathematics*, by Dr. Hillel Poritsky, General Electric Company.

This paper was concerned with the mathematics which is found useful in an industrial concern. A general discussion of applied versus pure mathematics was given, typical problems in various fields of applied mathematics were presented, and methods and tools used in their solution were briefly discussed.

C. W. MUNSHOWER, *Secretary*

MAY MEETING OF THE WISCONSIN SECTION

The sixteenth annual spring meeting of the Wisconsin Section of the Mathematical Association of America was held at Beloit College, Beloit, Wisconsin, on Saturday, May 8, 1948. Professor R. C. Huffer, Chairman of the Section, presided at the morning and afternoon sessions.

There were seventy-three in attendance, including the following thirty members of the Association: K. J. Arnold, R. H. Bardell, Leon Battig, L. J. Berner, A. C. Berry, W. W. Bigelow, R. H. Bing, H. J. Cohen, B. H. Colvin, H. H. Conwell, H. P. Evans, Harold Glander, E. G. Harrell, W. W. Hart, R. C. Huffer, W. R. Jarmain, J. F. Kenney, R. E. Langer, Sister Mary Felice, J. R. Mayor A. C. Moeller, H. P. Pettit, O. W. Rechard, R. D. Specht, Abraham Spitzbart, P. L. Trump, E. C. Varnum, J. I. Vass, R. D. Wagner, and Louise A. Wolf.

At the business meeting held during the afternoon session the following officers were elected for the coming year; Chairman, A. C. Berry, Lawrence College; Secretary, Louise A. Wolf, University of Wisconsin in Milwaukee; Program Committee, Sister Mary Felice, Mount Mary College, Elli Otteson, Eau Claire High School, R. D. Specht, University of Wisconsin. It was announced that R. H. Bardell, University of Wisconsin in Milwaukee, had been elected Sectional Governor from Wisconsin.

The afternoon program included reports by representatives of the various educational institutions on the status of mathematical education in Wisconsin. This was followed by the passing of a resolution that the Chairman appoint a committee to be charged with the duty of seeing to the distribution of the pamphlet on guidance throughout the schools of the state.

The following papers were presented:

1. *Fourier series*, by Professor R. E. Langer, University of Wisconsin.

The paper purported to review the growth and development of the ideas which are the nucleus of the theory of the Fourier series, and to project this development back upon the historical and biographical environment in which it took place. Beginning with the analysis of the vibrating elastic string, the contrasting approaches to the problem, and the conflicting ideas, assumptions, and implications which culminated in the D'Alembert-Euler-Bernoulli controversy (about 1750) were set forth. Euler's determination of the coefficients (about 1775) was explained, and the work of Fourier (soon after 1800) was discussed, both as to its formal character, and as to its novel interpretations and revolutionary consequences.

In conclusion, a brief outline of developments which owed their inceptions to the Fourier series and which proved to be of fundamental significance in the shaping of analysis generally was given. This included references to Dirichlet's distinction between the absolute and conditional convergence of an infinite series, the clarification of the notion of a function, the discovery of uniform convergence, the definition of the Riemann integral, the discovery of the continuous non-differentiable curve, Cantor's conceptions of the theory of point sets and of trans-finite numbers, the theory of integral equations, the theory of differential boundary value problems, etc.

2. *The use of nomograms in industry*, by E. C. Varnum, Mathematician, Barber-Colman Company, Rockford, Illinois.

After a brief review of nomographic theory based on determinantal methods, several useful nomograms constructed in the development department of a large industrial plant were shown, along with their applications to electrical and mechanical problems.

LOUISE A. WOLF, *Secretary*

CALENDAR OF FUTURE MEETINGS

Joint Meeting with American Society for Engineering Education, Troy, New York, June 20–21, 1949.

Thirty-first Summer Meeting, Boulder, Colorado, August 29–30, 1949.

Thirty-third Annual Meeting, New York City, December 30, 1949.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, May 7, 1949

ILLINOIS, Bradley University, Peoria, May 13–14, 1949.

INDIANA, University of Notre Dame, May 7, 1949

IOWA, Drake University, Des Moines, April 15–16, 1949

KANSAS, Manhattan, April 2, 1949

KENTUCKY

LOUISIANA-MISSISSIPPI, University of Mississippi, Oxford, April 8–9, 1949

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Virginia, Charlottesville, Spring, 1949

METROPOLITAN NEW YORK, Brooklyn College, April 9, 1949

MICHIGAN, Wayne University, Detroit, April 2, 1949

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NEBRASKA, Lincoln, May, 1949

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OHIO, Ohio State University, Columbus, April 2, 1949

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PACIFIC NORTHWEST, Oregon State College, Corvallis, March 25–26, 1949

PHILADELPHIA, Haverford College, November 26, 1949

ROCKY MOUNTAIN, Colorado School of Mines, Golden, April, 1949

SOUTHEASTERN, University of Alabama, University, March 18–19, 1949

SOUTHERN CALIFORNIA, John Muir Junior College, Pasadena, March 12, 1949

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TEXAS, Denton, Spring, 1949

UPPER NEW YORK STATE, University of Buffalo, April 30, 1949

WISCONSIN, Lawrence College, Appleton, May 14, 1949

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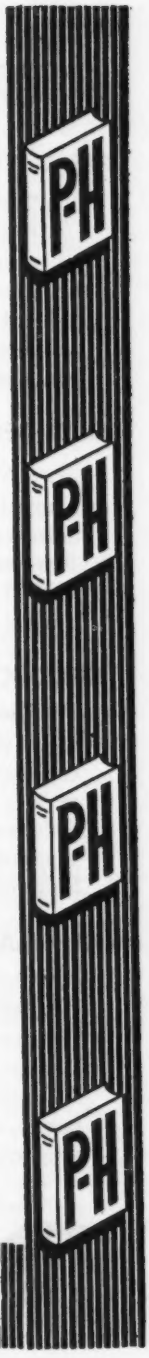

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